On a problem of discrete mathematics which has its origin from isolated surface singularities with $\mathbb{C}^*$-action

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1. Introduction

Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers. In this paper we think over the invariants of surface singularities with $\mathbb{C}^*$-action by applying discrete mathematical methods. So, we will give a brief summary of $\mathbb{C}^*$ singularities.

Definition 1. *A polynomial $f(x, y, z)$ over $\mathbb{C}$ is weighted homogeneous of type $(a, b, c; h)$ where a, b, c and h are positive integers, if it can be expressed as a linear combination of monomials $x^i y^j z^k$ such that $ai + bj + ck = h$.*

We consider a weighted homogeneous polynomial of type $(a, b, c; h)$ with $\gcd(a, b, c) = 1$ which defines a surface on $\mathbb{C}^3$ such that it has an isolated singularity at the origin, because this isolated singularity admits good $\mathbb{C}^*$-action. According to [**4**], we call the notation $(a, b, c; h)$ used in Definition 1 a system of weights for answering the next question. Which type of a system of weights corresponds to singularities with good $\mathbb{C}^*$-action?

Definition 2. *Let $(a, b, c; h)$ be a system of weights, i.e., a, b, c and h are positive integers and $h > \max(a, b, c)$ is also assumed. A system of weights $(a, b, c; h)$ is regular if the rational function in T*

$$T^{a+b+c} \frac{(T^{h-a} - 1)(T^{h-b} - 1)(T^{h-c} - 1)}{(T^a - 1)(T^b - 1)(T^c - 1)} \tag{1}$$

can be simplified into a polynomial. In additional, if it has the property $\gcd(a, b, c) = 1$ then it is called reduced.

Lemma 1. *A reduced regular system of weights $(a, b, c; h)$ has the following properties.*

(i) *For any two elements α and β in the set of weights $\{a, b, c\}$, $\gcd(\alpha, \beta)$ divides the degree h .*

(ii) *Each of the weights divides at least one from among $h - a, h - b$ and $h - c$.*

(iii) *Any of the weights is not larger than $\frac{h}{2}$.*

Proposition 1 ([**4**]). *Suppose $(a,b,c;h)$ is a reduced regular system of weights. By choosing suitable coefficients, a weighted homogeneous polynomial f of type $(a,b,c;h)$ has an isolated singularity at the origin.*

See the reference [**4**] for more details. We may assume $a \leq b \leq c$. Now, the dual graph of the canonical resolution of the singularity described above is star-shaped which corresponds to the central curve and some rational curves if there exists. More precisely, the geometrical invariants of exceptional curves are completely determined by the reduced regular system of weights $(a,b,c;h)$. In particular, the following proposition gives us information about the central curve, what we should focus on.

Proposition 2 ([**2**][**3**]). *Suppose that $(a,b,c;h)$ is a reduced regular system of weights. Then the genus of the central curve g is as follows:*

$$g = \frac{1}{2}\left\{\frac{h^2}{abc} - h\frac{(a,b)}{ab} - h\frac{(b,c)}{bc} - h\frac{(c,a)}{ca} + \frac{(a,h)}{a} + \frac{(b,h)}{b} + \frac{(c,h)}{c} - 1\right\} \tag{2}$$

where the notation (a,b) implies $\gcd(a,b)$ *as usual.*

2. A proof of the genus formula

Wagreich explains the correspondence between the isolated surface singularities with $\mathbb{C}^*$-action and the graded rings of finite type in [**6**]. Thus the genus formula, which we have showed in Proposition 2, is proved in the different ways. In [**3**], the formula is obtained from the general theory of the graded ring. We will recall the outline of the proof briefly.

Definition 3. *Suppose that a ring R has the direct sum decomposition, i.e. $R = \oplus_{i=-\infty}^{\infty} R_i$. If $xy \in R_{i+j}$ for $x \in R_i$ and $y \in R_j$, R is called a graded ring.*

A graded ring we treat is a ring of finite type over $\mathbb{C}$ with $R_i = 0$ for all $i < 0$.

Definition 4. *Let R be a graded ring of finite type with $R_0 = \mathbb{C}$. The Paincaré power series of R is $\wp(T) = \sum_{i=0}^{\infty} a_i T$ where a_i is $\dim_{\mathbb{C}} R_i$.*

For a reduced regular system of weights $(a,b,c;h)$, if $f(x,y,z)$ is the defining equation of a surface which has an isolated singularity at the origin, the corresponding graded ring R_f is $\mathbb{C}[x,y,z]/I$ where I is the ideal generated by f. Note the weights a, b and c induce the natural grading on this ring.

Proposition 3. *The Paincaré power series $\wp(T)$ of the graded ring R_f as* *above is given by*

$$\wp(T) = \frac{(T^h - 1)}{(T^a - 1)(T^b - 1)(T^c - 1)}. \tag{3}$$

There exists a polynomial $p(x)$, which is called Hilbert polynomial, so that $p(i) = dim_{\mathbb{C}} R_{di}$ for $i \gg 0$ where $d = abc$. By the fact that the genus g is equal to $1 - p(0)$, the formula desired is given.

Moreover, Prof. Saito insists on another way in order to find the genus in [**5**]. We will show it in the next section.

3. A restricted partition function

First, we need to prepare several terms and notations. Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers. As usual, $\lfloor x \rfloor$ is the floor function for real number x and $\{x\}$ sometimes means $x - \lfloor x \rfloor$.

Definition 5. *Suppose A is a set of k positive integers $\{a_1, a_2, \ldots, a_k\}$. Then the function $p_A(n)$ defined for non-negative integers is a restricted partition function if*

$$p_A(n) = \#\left\{(x_1, x_2, \cdots, x_k) \in \mathbb{Z}^k_{\geq 0} \mid a_1x_1 + a_2x_2 + \cdots + a_kx_k = n\right\}. \tag{4}$$

We also use the similar notation as follows:

$$\overset{\circ}{p}_A(n) = \#\left\{(x_1, x_2, \cdots, x_k) \in \mathbb{N}^k \mid a_1x_1 + a_2x_2 + \cdots + a_kx_k = n\right\}. \tag{5}$$

The generating function for $\{p_A(n)\}_{n\geq 0}$ is well-known as

$$\sum_{n\geq 0} p_A(n)T^n = \prod_{i=1}^{k} \frac{1}{1 - T^{a_k}}. \tag{6}$$

In case of $A = \{a, b\}$, the next theorem is useful.

Theorem 1 (Popoviciu [**1**]). *Suppose two positive integers a and b are relatively prime. For any positive integer n,*

$$p_{\{a,b\}}(n) = \frac{n}{ab} - \left\{\frac{b^{-1}n}{a}\right\} - \left\{\frac{a^{-1}n}{b}\right\} + 1 \tag{7}$$

where $aa^{-1} \equiv 1 \pmod b$ and $bb^{-1} \equiv 1 \pmod a$.

In [**1**], you can see the proof used such a general method that the generating rational function for a sequence is decomposed as a partial fraction. If $d = (a, b)$ divides n then for $a' = \frac{a}{d}, b' = \frac{b}{d}$ and $n' = \frac{n}{d}$ we can apply the Theorem 1 because of $p_{\{a,b\}}(n) = p_{\{a',b'\}}(n')$. In other case, it is clear that $p_{\{a,b\}}(n) = 0$.

Moreover, if we choose a suitable solution $(x_0, y_0) \in \mathbb{Z}^2$ for the equation $ax + by = n$, the equation $ax_0 + by_0 = n$ induces $x_0 \equiv a'^{-1}n' \pmod{b'}$ and $y_0 \equiv b'^{-1}n' \pmod{a'}$. Since any solution (x_k, y_k) can be expressed as $x_k = x_0 + bl$, $y_k = y_0 - al$ by a suitable integer l, we get the next Corollary with easy.

Corollary 1. *Suppose $d = (a, b)$ divides n. So, there are infinitely many solutions $(x_k, y_k) \in \mathbb{Z}^2$ for the equation $ax + by = n$. Then it is hold that*

$$p_{\{a,b\}}(n) = \frac{dn}{ab} - \left\{\frac{dy_k}{a}\right\} - \left\{\frac{dx_k}{b}\right\} + 1. \tag{8}$$

Next, we will see Popoviciu's Theorem from the viewpoint of elementary geometry through some examples.

Example 1. Let a, b and n be $2, 3$ and 17 respectively, so we have a solution $(x_0, y_0) = (1, 5)$ for $2x + 3y = 17$. By Corollary 1, we find

$$p_{\{2,3\}}(17) = \frac{17}{6} - \left\{\frac{5}{2}\right\} - \left\{\frac{1}{3}\right\} + 1 = 3. \tag{9}$$

Even though we choose a solution $(x_0, y_0) = (-2, 7)$, we can get the same answer from $\left\{\frac{5}{2}\right\} = \left\{\frac{7}{2}\right\} = \frac{1}{2}$ and $\left\{\frac{1}{3}\right\} = \left\{\frac{-2}{3}\right\} = \frac{1}{3}$. Now, three positive solutions which the equation (9) indicates are regarded as the lattice points P, Q and R on the line $2x + 3y = 17$ in Figure 1. Here, we adopt the distance between the two points P and Q, which are next to each other, as a unit. It means that the lengths of $\overline{AP}$ and $\overline{RB}$ are $\frac{1}{3}$ and $\frac{1}{2}$ respectively by this measure. As result, $p_{\{2,3\}}(17) - 1$ is consistent with the length of $\overline{PR}$ by this measure, that is two units.

In general, the above observation is true. The measure can be induced by a unit caused from the lattice points P and Q on the line $ax + by = n$ which are next to each other. Then $\frac{dn}{ab}$ implies the length of $\overline{AB}$ by this measure. Also, let $P(x_0, y_0)$ be the nearest lattice point on the right of A, and the length of $\overline{AP}$ is $\{\frac{dx_0}{b}\}$. Similarly, let $R(x_1, y_1)$ be the nearest lattice point on the left of B, and the length of $\overline{RB}$ is $\{\frac{dy_1}{a}\}$. So P is located on $\frac{dn}{ab} - \{\frac{dy_1}{a}\} - \{\frac{dx_0}{b}\}$ units from R where the positive direction is right side. Therefore in addition one to this value, we obtain the number of the lattice points on $\overline{AB}$.

Example 2. Let a, b and n be $3, 7$ and 5 respectively, a solution (x_0, y_0) for

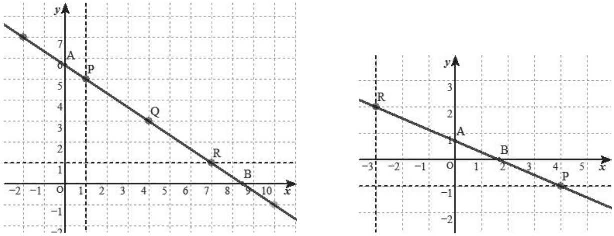

Figure 1.

$3x+7y=5$ may be $(-3,2)$ or $(4,-1)$. We get $p_{\{3,7\}}(5) = \frac{5}{21} - \left\{\frac{2}{3}\right\} - \left\{\frac{4}{7}\right\} + 1 = 0$.

We can also give a similar observation for $A = \{a,b,c\}$. Although $p_A(h)$ is the number of lattice points on the domain $\{(x,y,z) \in H \mid x,y,z \geq 0\}$ in the plane $H : ax+by+cz = h$, we will treat the points which are projected to the xy-plane, occasionally the yz-plane or the xz-plane, in order to compute them with easy.

Definition 6. *For any lattice point* $(x_0, y_0, z_0) \in \mathbb{Z}^3$ *on a plane, we call* (x_0, y_0)*, which may be* (y_0, z_0) *or* (x_0, z_0)*, a projective lattice point.*

Example 3. For a reduced regular system of weights $(2,3,9;18)$, Figure 2 shows the projective lattice points corresponding to the plane $2x+3y+9z=18$ and $p_{\{2,3,9\}}(18) = 7$. Moreover, we can find Pick's theorem.

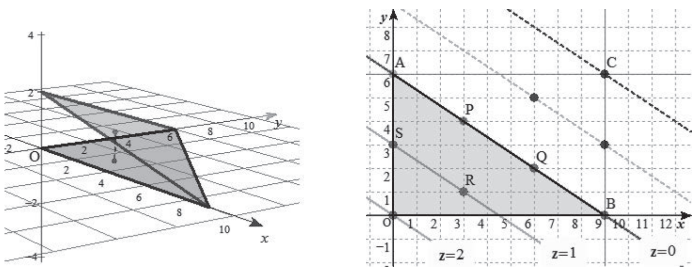

Figure 2.

Theorem 2 (Pick [**1**]). *For integral convex polytope* P*,* $i(\mathrm{P})$ *and* $b(\mathrm{P})$ *denote the number of the lattice points in the interior of* P *and that on the boundary of* P *respectively. Then the area of* P *is equal to* $i(\mathrm{P}) + \frac{1}{2}b(\mathrm{P}) - 1$.

Consider the coordinate system which has the fundamental parallelogram $PQRS$ as a unit area, and the area of the colored triangle Δ is 3. On the other hand, we can think the projective lattice points as the "ordinary" lattice points, so there are 7 lattice points on Δ with including 6 ones on the boundary of Δ. Thus we get $i(\Delta)+\frac{1}{2}b(\Delta)-1=3$. In addition to say, the area of the rectangle $OACB$ is double the area of Δ and it is equal to $2i(\Delta)+b(\Delta)-2=p_{\{a,b,c\}}(h)+\overset{\circ}{p}_{\{a,b,c\}}(h)-2$ where a,b,c and h are $2,3,9$ and 18. We will discuss this observation in the general cases later.

Finally, we can mention the other aspect of the genus.

Proposition 4 (Saito [**5**]). *The genus g of the central curve determined by a reduced regular system of weights $(a,b,c;h)$ is equal to $p_{\{a,b,c\}}(\epsilon_0)$ where ϵ_0 is $h-a-b-c$, that is $g=\overset{\circ}{p}_{\{a,b,c\}}(h)$.*

We know the genus g is 1 by using the equation (2) for Example 3 and it is consistent with $\overset{\circ}{p}_{\{a,b,c\}}(h)$.

4. A main result

In this secton, we will directrly connect the genus formula in Proposition 2 with the formula in Proposition 4 by computing the corresponding lattice points.

First, we give some lemmas and notations for a reduced regular system of weights $(a,b,c;h)$. From now on, A denotes the set of weights $\{a,b,c\}$.

Lemma 2. *Let a and h be positive integers.*

$$p_{\{a\}}(h)=\frac{(a,h)}{a}-\left\{\frac{(a,h)}{a}\right\} \tag{10}$$

It is clear that $p_{\{a\}}(h)=1$ if and only if $a\mid h$, i.e., a divides h, but otherwise $p_{\{a\}}(h)=0$. When we consider any element α in the set A as a, δ_α denotes $\left\{\frac{(\alpha,h)}{\alpha}\right\}$ in the the equation (10).

By Lemma 1 (i), we apply Corollary 1 for any two elements α and β in the set A, then we have

$$p_{\{\alpha,\beta\}}(h)=\frac{(\alpha,\beta)h}{\alpha\beta}-\left\{\frac{(\alpha,\beta)y_0}{\alpha}\right\}-\left\{\frac{(\alpha,\beta)x_0}{\beta}\right\}+1 \tag{11}$$

where (x_0,y_0) is a suitable solution of $\alpha x+\beta y=h$. Also, $\delta_{\alpha,\beta}$ denotes $\left\{\frac{dy_0}{\alpha}\right\}+\left\{\frac{dx_0}{\beta}\right\}$ in the the equation (11) where $d=(\alpha,\beta)$, and we get the folloing lemma from Lemma 1 (ii).

Lemma 3. *For a reduced regular system of weights $(a, b, c; h)$, the following properties are hold.*

(i) $\alpha \mid h \Rightarrow \delta_{\alpha,\beta} = \{\frac{(\alpha,\beta)h}{\alpha\beta}\}$ *and* $\delta_{\alpha,\gamma} = \{\frac{(\alpha,\gamma)h}{\alpha\gamma}\}$

(ii) $\gamma \mid h - \alpha \Rightarrow \delta_\gamma = \{\frac{(\alpha,\gamma)}{\gamma}\}$ *and* $\delta_{\alpha,\gamma} = \delta_\gamma + \{\frac{(\alpha,\gamma)(h-\alpha)}{\alpha\gamma}\}$

Under the condition $\gamma \mid h - \alpha$,

(a) $\alpha \mid h \Rightarrow \delta_{\alpha,\gamma} = \delta_\gamma$

(b) $\beta \mid h \Rightarrow \delta_{\beta,\gamma} = \left\{\frac{h}{\beta\gamma}\right\}$

Proof. (i) Choose $(x_0, y_0) = (\frac{h}{\alpha}, 0)$ and $(x_0, z_0) = (\frac{h}{\alpha}, 0)$ for the equations $h = \alpha x + \beta y$ and $h = \alpha x + \gamma z$ respectively.
(ii) Since $(\alpha, \gamma) \mid h$ by Lemma 1 (i), it follows from $(\alpha, \gamma) = (\gamma, h)$.
(a) By the assumption, we have the expression $h = \alpha + \gamma y_0 = \alpha x_1$, so we get $\delta_{\alpha,\gamma} = \{\frac{(\alpha,\gamma)}{\gamma}\} = \delta_\gamma$. **(b)** It is clear by the next remark. □

Remark 1. If $\gamma \mid h - \alpha$ then β and γ are relatively prime. Because $(\beta, \gamma) \mid h$ by Lemma 1 (i) and then $(\beta, \gamma) \mid \alpha$, but $(a, b, c; h)$ is reduced, i.e., $(a, b, c) = 1$,

By Definition 5, the next lemma is hold.

Lemma 4. *Let A be a set of three positive integers and h be a positive integer.*

$$p_A(h) = \overset{\circ}{p}_A(h) + \sum_{\{\alpha,\beta\}\subset A} p_{\{\alpha,\beta\}}(h) - \sum_{\alpha\in A} p_{\{\alpha\}}(h) \tag{12}$$

Next, we rewrite the genus formula in Proposition 2 by using both equations (10) and (11) as follows:

$$\begin{aligned} 2g &= \frac{h^2}{abc} - \sum_{\{\alpha,\beta\}\subset A} \left(p_{\{\alpha,\beta\}}(h) + \delta_{\alpha,\beta}\right) + \sum_{\alpha\in A} \left(p_{\{\alpha\}}(h) + \delta_\alpha\right) + 2 \\ &= \left\lfloor \frac{h^2}{abc} \right\rfloor - \sum_{\{\alpha,\beta\}\subset A} p_{\{\alpha,\beta\}}(h) + \sum_{\alpha\in A} p_{\{\alpha\}}(h) - \delta + 2 \end{aligned} \tag{13}$$

where $\delta = \sum_{\{\alpha,\beta\}\subset A} \delta_{\alpha,\beta} - \sum_{\alpha\in A} \delta_\alpha - \left\{\frac{h^2}{abc}\right\}$.

We should recognize δ is integer. Now, comparing the equation (12) to the equation (13) give us the next result.

Proposition 5. *Suppose $(a,b,c;h)$ is a reduced regular system of weights. If $r=\#\{\alpha\in A\mid \alpha\nmid h\}$ then the following is hold.*

$$p_A(h)+\overset{\circ}{p}_A(h)=\left\lfloor\frac{h^2}{abc}\right\rfloor-\delta+2 \quad where \quad \begin{cases} \delta=0 & for \quad r=0,1 \\ 0\le\delta\le r-1 & for \quad r=2,3 \end{cases} \tag{14}$$

Finally, this proposition gives what we want.

Corollary 2. *The genus g is equal to $\overset{\circ}{p}_A(h)$.*

Proof. We have only to examine the equation (14) is true for cases I $\sim$ IV, which are classified in order from $r=0$ to 3.

As in section 3, we treat projective lattice points. Moreover, for the lattice point (x_0,y_0,z_0) on the plane $\alpha x+\beta y+\gamma z=h$, where $\{\alpha,\beta,\gamma\}=A=\{a,b,c\}$, we will express the corresponding projective lattice point as $[x_0,y_0,z_0]$. Also, the line $\alpha x+\beta y=h$ can be expressed as $z=0$. So, projective lattice points are only on the line $z=k(\alpha,\beta)\gamma$ where k is an integer, because of $(\alpha,\beta,\gamma)=1$.

Case I. Since all weights a,b and c divide h, we see that $\frac{h^2}{abc}$ is integer by Lemma 1 (i) and $(a,b,c)=1$. In Figure 3, consider the fundamental parallelogram $APRS$

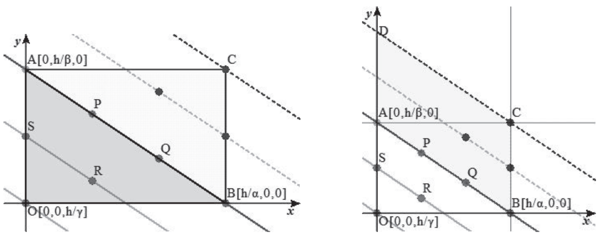

Figure 3.

as a unit area, which means the length of $\overline{AP}$ and that of $\overline{AS}$ are regarded as units respectively. So, the length of $\overline{AB}$ may be $\frac{(\alpha,\beta)h}{\alpha\beta}$, and then $\frac{(\alpha,\beta)h}{\alpha\beta}\cdot\frac{h}{(\alpha,\beta)\gamma}=\frac{h^2}{abc}$ means the area of the rectangle $OACB$ by this measure. Now, let Δ denote the triangle OAB. By noting that the projective lattice points are regarded as ordinary ones, it is clear that $p_A(h)$ implies the number of projective points on Δ. Also, we see that

$$i(\Delta)=\overset{\circ}{p}_A(h) \quad \text{and} \quad b(\Delta)=p_A(h)-\overset{\circ}{p}_A(h).$$

Thus, by applying Theorem 2 for the triangle ABO or the rectangle $OACB$, we obtain the equation (14).

Case II We assume that $\alpha \nmid h$ and $\beta,\ \gamma \mid h$ and may be $\alpha \mid h-\gamma$. First, it's clear that $\delta_\beta = \delta_\gamma = 0$ and we have $\delta_{\beta,\gamma} = 0$ from Corollary 1 or Figure 1. Next, we get $\delta_{\alpha,\gamma} = \delta_\alpha = \frac{1}{\alpha}$ and $\delta_{\alpha,\beta} = \left\{\frac{h}{\alpha\beta}\right\}$ form Lemma 3 (ii). As we see $\frac{(\beta,\gamma)h}{\beta\gamma},\ \frac{h-\gamma}{(\beta,\gamma)\alpha} \in \mathbb{Z}$, we obtain $\delta = 0$ by noting $\left\{\frac{h^2}{\alpha\beta\gamma}\right\} = \delta_{\alpha,\beta}$ from

$$\frac{(\beta,\gamma)h}{\beta\gamma} \cdot \frac{h-\gamma}{(\beta,\gamma)\alpha} = \frac{h^2}{\alpha\beta\gamma} - \frac{h}{\alpha\beta} \in \mathbb{Z}, \tag{15}$$

Finally, we adopt the fundamental parallelogram as the unit area like Case I and examine the equation (15) by using Figure 4. The left side and the first term

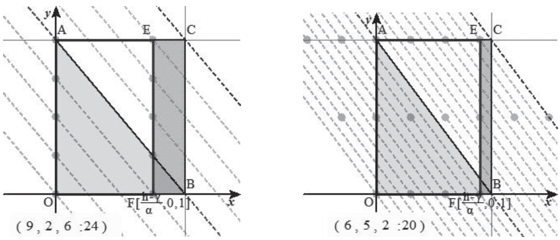

Figure 4.

of the right side imply the areas of the rectangles $OAEF$ and $OACB$ respectively.

By considering about $\overline{OA}$ as like as Figure 1, we see that $\left\lfloor\frac{h}{\alpha\beta}\right\rfloor$ means the number of the lattice points except for A on $\overline{OA}$. Notice that $\left\lfloor\frac{h^2}{abc}\right\rfloor$ is the sum of the area of $OAEF$ and $\left\lfloor\frac{h}{\alpha\beta}\right\rfloor$, and also that there are no lattice points in the interior of the triangle ABF . By applying Theorem 2 for the triangle OAF. we obtain $\left\lfloor\frac{h^2}{abc}\right\rfloor = 2i(\Delta) + b(\Delta) - 2$ where Δ is same as Case I.

Case III We assume that $\alpha \mid h$ and $\beta,\ \gamma \nmid h$, so we may take the following cases (i) $\sim$ (iii). For each case, δ can be easily calculated by Lemma 3 and Remark 1, so we will omit these proofs. As a result, we find out that the integer δ is 0 or 1.

(i) $\beta,\ \gamma \mid h-\alpha \ \Rightarrow\ \alpha \mid h,\ \beta \mid h-\alpha,\ \gamma \mid h-\alpha$

If β and γ divide neither $h-\beta$ nor $h-\gamma$, we may suppose $\alpha \mid h-\beta$ by Definition 2. Then we get $(\alpha,\beta) = \alpha,\ (\alpha,\gamma) = 1,\ (\beta,\gamma) = 1$ and we have

$$\delta_{\alpha,\beta} = \delta_\beta = \frac{\alpha}{\beta},\ \delta_{\alpha,\gamma} = \delta_\gamma = \frac{1}{\gamma},\ \delta = \delta_{\beta,\gamma} - \left\{\frac{h^2}{\alpha\beta\gamma}\right\}.$$

We choose the projection to the yz-plane like Figure 5 in order to describe projective lattice points. Of course, the origin O is a lattice point and there are the lattice points $H[1, 0, \frac{h-\alpha}{\gamma}]$, $T[1, \frac{h-\alpha}{\beta}, 0]$, $G[\frac{h-\beta}{\alpha}, 1, 0]$ on the y, z axes. Moreover,

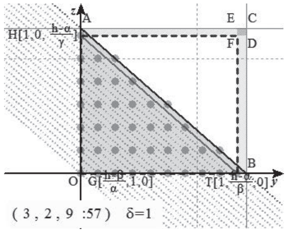

Figure 5.

the number of the lattice points on $\overline{HT}$ and that of $\overline{AB}$ are as follows:

$$p_{\{\beta,\gamma\}}(h-\alpha) = \frac{h-\alpha}{\beta\gamma} + 1,\ p_{\{\beta,\gamma\}}(h) = \frac{h}{\beta\gamma} - \delta_{\beta,\gamma} + 1 = p_{\{\beta,\gamma\}}(h-\alpha) - \delta.$$

Here we have $\left\lfloor \frac{h}{\beta\gamma} \right\rfloor = \frac{h-\alpha}{\beta\gamma}$ and $\left\{ \frac{h}{\beta\gamma} \right\} = \frac{\alpha}{\beta\gamma}$ from $\alpha < \beta$. So we see $\delta = \delta_{\beta,\gamma} - \frac{\alpha}{\beta\gamma}$ and it is clear that $\delta = 0$ or 1. Let n be $p_{\{\beta,\gamma\}}(h-\alpha)$, which is the number of the lattice points on $\overline{HT}$. Similarly, the number of the units on $\overline{OA}$ and $\overline{OB}$, which is denoted as l and m respectively, are as follows:

$$l = p_{\{\alpha,\gamma\}}(h) - 1 = \frac{h-\alpha}{\alpha\gamma}, \quad m = p_{\{\alpha,\beta\}}(h) - 1 = \frac{(\alpha,\beta)(h-\alpha)}{\alpha\beta} = \frac{h-\alpha}{\beta}.$$

Now, the rectangle $OACB$ is divided into four ones, as if we have done to compute the lattice points about Case II. In fact,

$$\frac{h^2}{\alpha\beta\gamma} = \frac{h}{\alpha\gamma} \cdot \frac{h}{\beta} = \left(l + \frac{1}{\gamma}\right) \cdot \left(m + \frac{\alpha}{\beta}\right) = lm + \frac{m}{\gamma} + \frac{l\alpha}{\beta} + \frac{\alpha}{\beta\gamma}.$$

The right-side means the areas of $OHFT, AEFH, FDBT$ and $ECDF$ in order from left. Though we have already known $\left\{ \frac{h^2}{\alpha\beta\gamma} \right\} = \frac{\alpha}{\beta\gamma}$, we see it form $\frac{m}{\gamma} = \frac{l\alpha}{\beta} = \frac{h-\alpha}{\beta\gamma} = n - 1$. Let Δ' be the triangle OHT, and we have $lm = 2i(\Delta') + b(\Delta') - 2$ by Theorem 2. Thus, we obtain the result desired

$$\begin{aligned} \left\lfloor \frac{h^2}{\alpha\beta\gamma} \right\rfloor + 2 &= lm + 2n = (i(\Delta') + n - 2) + (i(\Delta') + b(\Delta') + n) \\ &= \overset{\circ}{p}_A(h) + p_A(h) + \delta. \end{aligned}$$

(ii) $\beta,\ \gamma \nmid h-\alpha\ \Rightarrow\ \alpha \mid h,\ \beta \mid h-\gamma,\ \gamma \mid h-\beta$

It is easy to know $(\alpha,\beta)=(\alpha,\gamma)=1$ and we see $(\beta,\gamma)<\min(\beta,\gamma)$ by Lemma 1. Here, we notice $\delta_{\beta,\gamma}<1$ and put $d=(\beta,\gamma)$ and $l=p_{\{\beta,\gamma\}}(h)-1$.

$$\delta=\delta_{\alpha,\beta}+\delta_{\alpha,\gamma}-\left\{\tfrac{h^2}{\alpha\beta\gamma}\right\}=\left\{\tfrac{h}{\alpha\beta}\right\}+\left\{\tfrac{h}{\alpha\gamma}\right\}-\left\{\tfrac{h^2}{\alpha\beta\gamma}\right\}$$

We choose the projection to the yz-plane like Figure 6 again. So, there are at least three lattice points $H[0,\frac{h-\gamma}{\beta},1]$, $T[0,1,\frac{h-\beta}{\gamma}]$ on $\overline{AB}$ and the origin O.

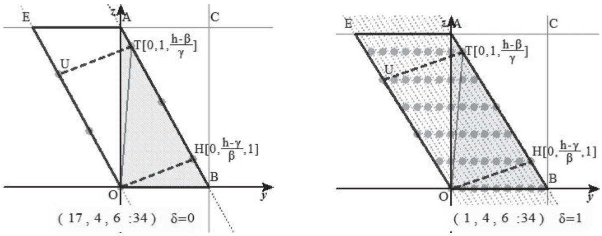

Figure 6.

First, we notice that both triangles OAT and OHB do not have any lattice point in their interiors. Moreover, there are no lattice points on $\overline{OT}$ except for both ends and it is same about $\overline{OH}$. Next, we focus on the lattice points on the rectangle $OUTH$, exclusive of O and T. So, they are arranged every l points along each line $x=dk$ for $k=0,\dots,\frac{h}{\alpha d}$. In addition to this, the number of lattice points on OUT is same as that on OTH. If we write the triangle OTH as Δ' then it is hold that $i(\Delta')=i(\Delta)$, which shows $lm=2i(\Delta)+l$ where $m=\frac{h}{\alpha d}$. On the other hand, we get

$$\frac{h^2}{\alpha\beta\gamma}=\frac{h}{\alpha}\cdot\frac{h}{\beta\gamma}=m\cdot(l+\delta_{\beta,\gamma})=lm+m\left(\frac{d}{\beta}+\frac{d}{\gamma}\right)=lm+\frac{h}{\alpha\beta}+\frac{h}{\alpha\gamma}.$$

Thus, we obtain

$$\begin{aligned}\left\lfloor\frac{h^2}{\alpha\beta\gamma}\right\rfloor+2&=lm+p_{\{\alpha,\beta\}}(h)+p_{\{\alpha,\gamma\}}(h)+\delta\\&=2i(\Delta)+p_{\{\beta,\gamma\}}(h)-1+p_{\{\alpha,\beta\}}(h)+p_{\{\alpha,\gamma\}}(h)+\delta\\&=\overset{\circ}{p}_A(h)+p_A(h)+\delta.\end{aligned}$$

(iii) $\beta \mid h-\alpha,\ \gamma\nmid h-\alpha \Rightarrow \alpha \mid h,\ \beta \mid h-\alpha,\ \gamma \mid h-\beta$

As we see $(\alpha,\gamma)=(\beta,\gamma)=1$ and $(\alpha,\beta)\geq 1$, put $d=(\alpha,\beta)$ and $l=p_{\{\alpha,\beta\}}(h)-1$. Here, we notice $l=\frac{d(h-\alpha)}{\alpha\beta}$ and $\delta_{\alpha,\beta}=\frac{d}{\beta}<1$. Also we put $\epsilon=\lfloor\delta_{\beta\gamma}\rfloor$ and then ε is 0 or 1, because of $0<\delta_{\beta,\gamma}<2$.

$$\delta=\delta_{\alpha,\gamma}+\delta_{\beta,\gamma}-\delta_{\gamma}-\left\{\tfrac{h^2}{\alpha\beta\gamma}\right\}=\left\{\tfrac{h}{\alpha\gamma}\right\}+\left\{\tfrac{h}{\beta\gamma}\right\}+\epsilon-\tfrac{1}{\gamma}-\left\{\tfrac{h^2}{\alpha\beta\gamma}\right\}.$$

In this case, the projection to the xy-plane like Figure 7 is useful. As there are three lattice points $T[0,1,\frac{h-\beta}{\gamma}$, $K[1,\frac{h-\alpha}{\beta},0]$ and $B[\frac{h}{\alpha},0,0]$, we can examine the triangle KTB as the case (i) or (ii).

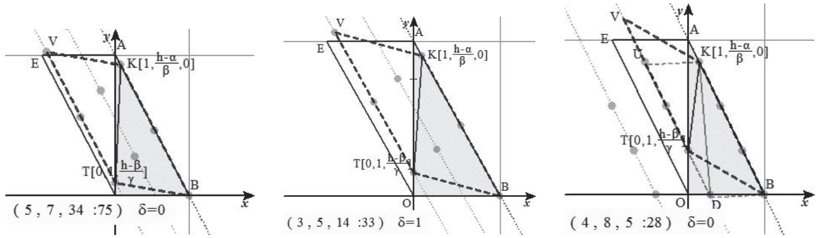

Figure 7.

But if $n=p_{\{\alpha,\gamma\}}(h)-1>0$, we may choose the triangle KDB where a point D on $\overline{OB}$ is not always projective lattice point but also ordinary one. It means that $D[\frac{\tilde{\beta}d}{\alpha},0,md]$ where we put $m=\left\lfloor\frac{h}{d\gamma}\right\rfloor$ and $\tilde{\beta}=\frac{h-md\gamma}{d}$. So we use Δ_1 for the former triangle and Δ_2 for the latter. Then we recognize $i(\Delta)=i(\Delta_1)=i(\Delta_2)$.

Here, we notice $m\geq\frac{h-\beta}{d\gamma}\in\mathbb{Z}$ and $\left\{\frac{h}{d\gamma}\right\}=\left\{\frac{\tilde{\beta}}{\gamma}\right\}$. Moreover if $n=0$ then we can show $\beta<d\gamma$. So we think under the assumption $\beta<d\gamma$ at first, whether n is 0 or not. Thus, we will carry out on the rectangle $KVTB$ what we have done in (ii). By $\beta=\tilde{\beta}d$, we get

$$\frac{h^2}{\alpha\beta\gamma}=\frac{dh}{\alpha\beta}\cdot\frac{h}{d\gamma}=\left(l+\frac{d}{\beta}\right)\cdot\left(m+\frac{\tilde{\beta}}{\gamma}\right)=lm+\frac{h}{\alpha\gamma}-\frac{1}{\gamma}+\frac{h}{\beta\gamma}.$$

It follows that

$$\left\lfloor\frac{h^2}{\alpha\beta\gamma}\right\rfloor+2=2i(\Delta)+p_{\{\alpha,\beta\}}(h)-1+p_{\{\alpha,\gamma\}}(h)+p_{\{\beta,\gamma\}}(h)+\delta.$$

Next, if $\beta>d\gamma$ then we will examine on the rectangle $KUDB$ because of $n>0$. We use the projection to the xz-plane like Figure 8 in order to compute. At first,

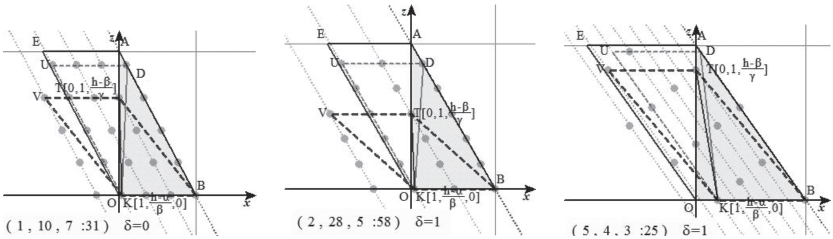

Figure 8.

we think the meaning of $l\frac{\tilde{\beta}}{\gamma}$ of the next equation

$$\frac{h^2}{\alpha\beta\gamma} = lm + \frac{l\tilde{\beta}}{\gamma} + \frac{h}{\beta\gamma}. \tag{16}$$

We recall $m = \left\lfloor \frac{h-\beta}{d\gamma} \right\rfloor + k$ where $k = \left\lfloor \frac{\beta}{d\gamma} \right\rfloor$. By $\beta \mid h-\alpha$, we put $M = \frac{h-\alpha}{\beta}$ and write $h = \alpha + \left(\tilde{\beta}d + d\gamma k\right) M$. Since $l = \frac{dM}{\alpha}$, we have

$$\frac{l\tilde{\beta}}{\gamma} = \frac{dM\tilde{\beta}}{\alpha\gamma} = \frac{h}{\alpha\gamma} - \frac{dkM}{\alpha} - \frac{1}{\gamma} = n + \delta_{\alpha,\gamma} - lk - \frac{1}{\gamma}. \tag{17}$$

Here, we notice $\left\lfloor \frac{l\tilde{\beta}}{\gamma} \right\rfloor = n - lk$ and $\left\{ \frac{l\tilde{\beta}}{\gamma} \right\} = \delta_{\alpha,\gamma} - \frac{1}{\gamma}$. If D is a lattice point then $n - lk = 1$, unless $n - lk = 0$. That is why the following equation is hold.

$$lm + n - lk = 2i(\Delta) + p_{\{\alpha,\beta\}}(h) + p_{\{\alpha,\gamma\}}(h) - 2. \tag{18}$$

Also the equation $lm+n-lk = l(m-k)+n$ shows the relation between the number of the lattice points on the rectangle $KUDB$ and that of the rectangle $KVTB$. Therefore we get the equation (14) by the equations (16), (17) and (18).

Example 4. For a reduced regular system of weight $(2, 28, 5; 58)$, we know $n = 5, l = 2$ from Figure 8. By $k = \left\lfloor \frac{28}{2\cdot 5} \right\rfloor = 2$, we get $n - kl = 5 - 2 = 1$. It means that D is a lattice point.

Case IV The last case has the property that any of weights a, b and c does not divide h, so we may think the following case (i) or (ii).

(i) $\alpha \mid h-\gamma$, $\beta \mid h-\gamma$, $\gamma \mid h-\alpha$

It is clear that $(\alpha,\beta) = (\beta,\gamma) = 1$. With regard to $d = (\alpha,\gamma)$, we see $d < min(\alpha,\gamma)$ like Case II (ii), so let $l = \left\lfloor \frac{dh}{\alpha\gamma} \right\rfloor$ and then we see $l = p_{\{\alpha,\gamma\}}(h) - 1$ by

$\delta_{\alpha,\gamma} = \frac{d}{\alpha} + \frac{d}{\gamma} < 1$, which means $h = \alpha + \gamma + \frac{\alpha\gamma}{d} l$.

Now, we choose the projection to the xz-plane like Figure 9. As there are three lattice points $G[\frac{h-\gamma}{\alpha}, 1, 0]$, $H[0, \frac{h-\gamma}{\beta}, 1]$ and $T[1, 0, \frac{h-\alpha}{\gamma}]$, we may examine the parallelogram $HUTG$ because of same reason as before.

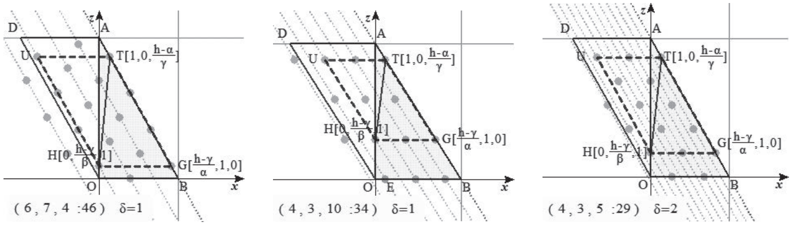

Figure 9.

First, it is trivial that $p_{\{\alpha,\beta\}}(h) \geq 1$, but we can show by the following well-known theorem. In general, for a set of positive integers $A = \{a_1, \ldots, a_n\}$, we define $g(a_1, \ldots, a_n) = \max\{n \in \mathbb{Z}_{\geq 0} \mid p_A(n) = 0\}$ and call it Frobenius number.

Theorem 3 (Frobenius). *Suppose two positive integers a and b are relatively prime. Then Frobenius number $g(a, b)$ is given by $ab - a - b$.*

By $(\alpha, \beta) = 1$ and $\alpha, \beta \mid h - \gamma$, we can write $h = \gamma + \alpha\beta k$ for a suitable positive integer k and get $p_{\{\alpha,\beta\}}(h) \geq 1$. Moreover, it may be $\delta_{\alpha\beta} > 1$, so we put $\epsilon = \lfloor \delta_{\alpha\beta} \rfloor$ where ϵ is 0 or 1. Since $p_{\{\alpha,\beta\}}(h - \gamma) = k + 1$, we see $p_{\{\alpha,\beta\}}(h) = k + 1 - \epsilon$ and $\delta_{\alpha\beta} = \epsilon + \frac{\gamma}{\alpha\beta}$.

$$\delta = \delta_{\alpha,\beta} + \delta_{\beta,\gamma} - \delta_\beta - \left\{ \frac{h^2}{\alpha\beta\gamma} \right\} = \frac{\gamma}{\alpha\beta} + \epsilon + \delta_{\beta,\gamma} - \frac{1}{\beta} - \left\{ \frac{h^2}{\alpha\beta\gamma} \right\}.$$

Here we put $m = \left\lfloor \frac{h-\gamma}{d\beta} \right\rfloor$ and have the equation

$$\frac{h^2}{\alpha\beta\gamma} = \frac{dh}{\alpha\gamma} \cdot \frac{h}{d\beta} = \left(l + \frac{d(\alpha+\gamma)}{\alpha\gamma} \right) \cdot \left(m + \frac{\gamma}{d\beta} \right) = lm + \frac{l\gamma}{d\beta} + \frac{h}{\alpha\beta} + \frac{h}{\beta\gamma}.$$

We see $\frac{l\gamma}{d\beta} = k - \frac{1}{\beta}$ from $h - \gamma = \alpha\beta k = \alpha + \frac{\alpha\gamma}{d} l$, so $lm + k$ is consistent with the number of the lattice points on $HUTG$ except for $\overline{HU}$, which is $2i(\Delta) + p_{\{\alpha,\gamma\}}(h)$. Thus we obtain

$$\left\lfloor \frac{h^2}{\alpha\beta\gamma} \right\rfloor + 2 = 2i(\Delta) + p_{\{\alpha,\beta\}}(h) + p_{\{\alpha,\gamma\}}(h) + p_{\{\beta,\gamma\}}(h) + \delta.$$

Lastly, it is clear that $0 \leq \delta - \epsilon < 2$. If $\epsilon = 0$ then $\alpha + d\beta = \gamma$ is satisfied by the observation of the arrangement for two lattice points, for example E and H on the center graph in Figure 9. Then $(\gamma.\beta, \alpha : h)$ is the same type as $(\alpha.\beta, \gamma : h)$ and $\delta_{\beta,\gamma} > 1$ is hold, because of $p_{\{\beta,\gamma\}}(h) = p_{\{\beta,\gamma\}}(h - \alpha) - 1$. Therefore we obtain δ is 1 or 2.

(ii) $\alpha \mid h - \gamma,\ \beta \mid h - \alpha,\ \gamma \mid h - \beta$

The case (ii) is the rest of the case (i), which means that we omit any reduced regular system of weights with the condition of the case (i). When we suppose that $\alpha < \beta < \gamma$, we may think about two cases by whether β divides $h - \gamma$ or $h - \alpha$. That is

(a) $\alpha \mid h - \gamma,\ \beta \mid h - \alpha,\ \gamma \mid h - \beta$ or

(b) $\alpha \mid h - \beta,\ \beta \mid h - \gamma,\ \gamma \mid h - \alpha$.

But we might be aware that there is no difference on the arrangements of lattice points with looking through Figure 10. Indeed, we can write $h = \gamma + l\alpha = \alpha + m\beta = \beta + n\gamma$ by a suitable positive integers l, m and n for the case (a). Here we note that l, m and n must be greater than 1. (See the point $D[l-1, 1-m, 1+n]$ in the left side of Figure 10 is located upper the line $z = \frac{h}{\gamma}$, which is character of this case.) Anyway, there are lattice points $G[l, 0, 1], H[1, m, 0]$ and $T[0, 1, n]$. Similarly, we can put $G[l, 1, 0], H[1, 0, n]$ and $T[0, m, 1]$ for the case (b).

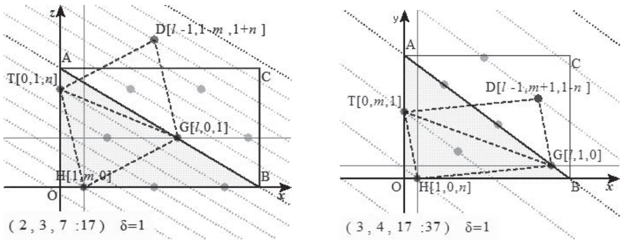

Figure 10.

In the case (ii), any two numbers of weights a, b and c are relatively prime.

(a) First, we see δ is like that

$$\delta = \left\{\frac{m}{\alpha}\right\} + \left\{\frac{n}{\beta}\right\} + \left\{\frac{l}{\gamma}\right\} - \left\{\frac{h^2}{\alpha\beta\gamma}\right\} \leq 2.$$

By the assumption, α, β and γ are expressed as follows:

$$\alpha = n\gamma + (1-m)\beta, \quad \beta = l\alpha + (1-n)\gamma, \quad \gamma = m\beta + (1-l)\alpha. \tag{19}$$

So we see $n < l$ and $n < m$ with easy. Moreover, we use some relations with weights, for instance $m\beta = ml\alpha + m(1-n)\gamma = \gamma - (1-l)\alpha$ and $(\alpha, \gamma) = 1$, and then we obtain

$$\alpha t = (n-1)m + 1, \quad \beta t = (l-1)n + 1, \quad \gamma t = (m-1)l + 1$$

where $t = p_{\{\alpha,\beta\}}(h)$. Because $p_{\{\beta,\gamma\}}(h) = 1$ by $h < \beta\gamma$, these equations can also be geometrically found from the triangle UVH or TUG where we put $V[n(1-l), nm+1, 0]$ and U divides $\overline{VT}$ in the ratio $n-1 : 1$. Next, we put $t_0 = \left\lfloor \frac{h}{\alpha\beta} \right\rfloor$

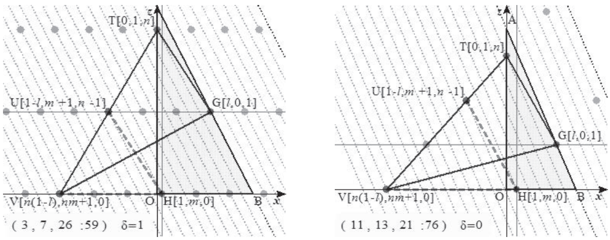

Figure 11.

and write $m = t_0\alpha + s$ where $0 < s < \alpha$. Since we have $\delta_{\alpha,\beta} = \frac{1}{\beta} + \frac{s}{\alpha} < 1$, we get $t_0 = t - 1$. It follows that

$$\begin{aligned} \frac{h^2}{\alpha\beta\gamma} &= \frac{h}{\alpha\beta} \cdot \frac{h}{\gamma} = \left(t_0 + \frac{1}{\beta} + \frac{s}{\alpha}\right) \cdot \left(n + \frac{\beta}{\gamma}\right) = nt_0 + \frac{n}{\beta} + \frac{ns}{\alpha} + \frac{h}{\alpha\gamma} \\ &= t_0 + \frac{s}{\alpha} + \frac{n}{\beta} + \frac{l}{\gamma} + \frac{(n-1)m+1}{\alpha} = t_0 + \frac{s}{\alpha} + \frac{n}{\beta} + \frac{l}{\gamma} + t \end{aligned}$$

Thus we obtain

$$\left\lfloor \frac{h^2}{\alpha\beta\gamma} \right\rfloor + 2 = t_0 + p_{\{\beta,\gamma\}}(h) + p_{\{\alpha,\gamma\}}(h) + \delta + p_{\{\alpha,\beta\}}(h).$$

If $h < \alpha\beta$ then H is the only lattice point on $\overline{OB}$. So we have $t = p_{\{\alpha,\beta\}}(h) = 1$ and $\delta_{\alpha,\beta} = \frac{1}{\beta} + \frac{m}{\alpha}$. As $h < \alpha\gamma$ and $h < \beta\gamma$ are also satisfied, so we get $p_A(h) = 3$ and $\overset{\circ}{p}_A(h) = 0$. Moreover, if $h > \alpha\beta$ then we see $n = 2$ by observing Figure 11, so we get $t_0 = 2\,\overset{\circ}{p}_A(h)$, Therefore, we obtain the equation (14).

(b) As we have decided l, m and n such that $h = \beta + l\alpha = \gamma + m\beta = \alpha + n\gamma$ for this case, we get

$$\delta = \left\{\frac{l}{\beta}\right\} + \left\{\frac{m}{\gamma}\right\} + \left\{\frac{n}{\alpha}\right\} - \left\{\frac{h^2}{\alpha\beta\gamma}\right\} \leq 2.$$

Moreover, we have the relations among weights as follows:

$$\alpha = m\beta + (1-n)\gamma, \quad \beta = n\gamma + (1-l)\alpha, \quad \gamma = l\alpha + (1-m)\beta.$$

Also, we see $m < l$ and $n < l$ by comparing to the equation (19). Now, we can write

$$\alpha t = (m-1)n + 1, \quad \beta t = (n-1)l + 1, \quad \gamma t = (l-1)m + 1, \tag{20}$$

for a suitable positive integer t. If $h < \beta\gamma$ then $p_{\{\beta,\gamma\}}(h) = 1$, so we see $t = p_{\{\alpha,\beta\}}(h)$.

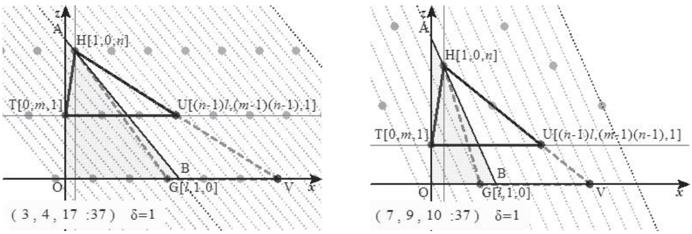

Figure 12.

Therefore we can give the proof as same as the case (a). Indeed, we have

$$\begin{aligned}\frac{h^2}{\alpha\beta\gamma} &= \frac{h}{\beta\gamma}\cdot\frac{h}{\alpha} = \left(\frac{1}{\beta}+\frac{m}{\gamma}\right)\cdot\left(1+\frac{n\gamma}{\alpha}\right) = \frac{1}{\beta}+\frac{m}{\gamma}+n\left(\frac{1}{\alpha}+\frac{l}{\beta}\right)\\ &= \frac{n}{\alpha}+\frac{m}{\gamma}+\frac{l}{\beta}+\frac{(n-1)l+1}{\beta} = \frac{n}{\alpha}+\frac{m}{\gamma}+\frac{l}{\beta}+t.\end{aligned}\tag{21}$$

Here if $t > 1$ then we see $n = 2$ and $\left\lfloor \frac{l}{\beta} \right\rfloor = t - 1 = 2\, \mathring{p}_A(h)$. If $t = 1$ then it is clear, so the equation (14) is hold for $h < \beta\gamma$.

Finally, we will examine the case with $h > \beta\gamma$. When we put $V[-1, 2m, 2-n]$, $\overline{VD}$ and $\overline{TG}$ intersect at F like Figure 13. Since $p_{\{\alpha,\beta\}}(h) \geq p_{\{\alpha,\gamma\}}(h) \geq 2$ and the triangles TVF and GDF are congruent, we see that $p_{\{\alpha,\beta\}}(h) = p_{\{\alpha,\gamma\}}(h) = p_{\{\beta,\gamma\}}(h) = 2$. Moreover, we obtain $t = 2\beta + 1$ by the equation (20) and the arrangement of the lattices.

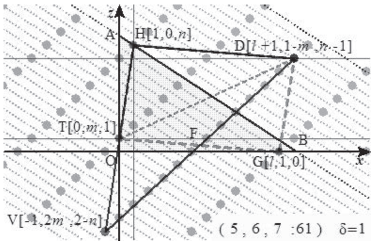

Figure 13.

Thus $t+1$ is the number of the lattice points on $\overline{VD}$, so $t-1 = 2 \overset{\circ}{p}_A(h)$. By the equation (21), we get

$$\left\lfloor \frac{h^2}{\alpha\beta\gamma} \right\rfloor + 2 = p_{\{\alpha,\gamma\}}(h) + p_{\{\beta,\gamma\}}(h) + p_{\{\alpha,\beta\}}(h) + t - 1 + \delta$$
$$= p_A(h) + \overset{\circ}{p}_A(h) + \delta.$$

□

Lastly, the author is interested in whether singularities or its minimal models for special cases like IV have something characteristic but particular combination of its weights.

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