Study on the mild solutions of some one-dimensional Keller-Segel systems expressed by Markovian semi-groups

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1 Introduction

In this thesis, the one-dimensional Keller-Segel system which has been posed as a mathematical and biological model by Keller and Segel [16] in 1970’s is considered. The system describes the phenomenon such that the cellular slime molds form an aggregation by the chemotaxis movement. Bellomo, Bellouquid, Tao and Winkler [7] give a general survey of the Keller-Segel system. In Hillen and Painter [14], there is a detailed exploration of variations of the Keller-Segel model. Here, we analyze an original model dealt in [14], which is denoted as follows:

\[
\begin{aligned}
\partial_t u &= \Delta u - \chi \nabla \cdot (u \nabla v) \quad \text{in } \Omega \times (0, \infty), \\
\partial_t v &= \Delta v - \gamma v + \alpha u \quad \text{in } \Omega \times (0, \infty), \\
(u(x, 0) &= \bar{u}(x) \geq 0, \\
v(x, 0) &= \bar{v}(x) \geq 0 \quad \text{in } \Omega,
\end{aligned}
\]

where \(\chi > 0, \alpha > 0\) and \(\gamma > 0\) are some given constants, and \(\Omega\) is a bounded or unbounded domain. The solutions \(u = u(x, t)\) and \(v = v(x, t)\) represent the cell density of the cellular slime molds and the cell concentration of the chemical substance that is released by the cellular slime molds at the position \(x\), and at time \(t\), respectively.

Since \(\partial_t u = \Delta u\) is the heat equation, and \(\partial_t v = -\nabla (u \cdot \chi \nabla v)\) is the equation of continuity by Euler, it follows that the first term on the right-hand side of the first equation of (KS) expresses the diffusion phenomenon, and the second term is read as the concentration phenomenon by which the cells move randomly. As a consequence, (KS) is composed by the terms of the concentration phenomenon and diffusion phenomenon.

After the pioneering work [16], (KS) has been generalized to several directions (see [14] for detailed exploring of various models). Many researchers are mainly interested in (KS) (see e.g. Viglialoro [30], Winkler [34] and Yahagi [36]), and the existence and uniqueness of the solution of (KS) are investigated (see e.g. Osaki and Yagi [25], Viglialoro [31]). In Marras, Vernier Piro and Viglialoro [20] and Mizoguchi and Winkler [21], there exist new results on the blow-up problems of (KS).

Recall that the original (KS) is a parabolic-parabolic type equation. We remark that there exist considerations on the following parabolic-elliptic type of the system.

\[
\begin{aligned}
\partial_t u &= \Delta u - \chi \nabla \cdot (u \nabla v), \\
0 &= \Delta v - \gamma v + \alpha u, \\
u(x, 0) &= \bar{u}(x) \geq 0, \\
v(x, 0) &= \bar{v}(x) \geq 0 \quad \text{in } \Omega,
\end{aligned}
\]

where \(\chi, \alpha, \gamma\) and \(\Omega\) are the same as in (KS). Nagai [23] considers (KS’) with a bounded domain \(\Omega\), and Sugiyama [29] considers it with the whole space \(\mathbb{R}^N\) \((N \geq 3)\). Yahagi [37] deals with the generalized Keller-Segel system which is based on (KS’) in the case of one-dimension.
This thesis is organized as follows. In Chapter 2, we give the brief explanations for the Keller-Segel system and the Brownian motion process. Some of numerical examples are also given.

In Chapters 3 and 4, as has been considered in Osaki and Yagi [24], we investigate the one-dimensional Keller-Segel system (KS) defined on a bounded interval with the Neumann boundary conditions (we denote it by the same symbol (KS)). We remark that this system given by (KS) is understood as a particular model where it is not affected by the concentration of a chemical substance, but is affected by the gradient of the concentration of a chemical substance.

Chapter 3 is devoted to show that a good estimate of the solution of (KS) around time zero is derived. Our analytical method for the consideration of the solution \((u, v)\) of (KS) is a stochastic analysis, by using the stochastic differential equation (SDE) driven by a standard Brownian motion. In Theorem 3.4, by using an expression of \((u, v)\) by means of expectation of functionals of the (SDE), we derive bounds for \((u, v)\) which gives a good estimate of \((u, v)\) around time zero. This theorem would be an application or modification of maximal principle in the usual analysis. However, it can be proved easily through the stochastic analytic methods. Figures 3 and 4 which are composed by using finite difference method, are visualizations of the results of Theorem 3.4. This chapter is based on Yahagi [35].

In Chapter 4, we focus on the case where the chemotaxis is small, and we analyze the asymptotic behavior of solutions to (KS) as the time development. We emphasize that the key method to prove our theorems is using the Fourier series. In Theorem 4.7, which we describe in Section 4.2, we show that the solutions of (KS) converge to some constants, as time tends to infinity, in the case of small chemotaxis. In general, it is well known that the properties of (KS) depend heavily on the dimensions. For the case of one-dimension, in Wang and Zhao [32] and Winkler [33], analogous results on the asymptotic behavior of the solutions to corresponding Keller-Segel systems, which are similar to but not same as the present (KS), are derived. In Cao [8], Cieślak, Laurençot and Morales-Rodrigo [9], the higher-dimensional analogue of the present (KS) is considered, and the corresponding results of the asymptotic behavior of the solutions are investigated. We also give a conjecture that if \(\chi\) in (KS) is sufficiently small, then the solutions of (KS) converge to some constants as time tends to infinity. In addition, we give some numerical simulations. To prove our conjecture is the future subject. This chapter is based on Yahagi [36].

In Chapter 5, we generalize the above (KS') which has uniform elliptic operators having variable coefficients. We call our system as a generalized Keller-Segel system (GKS) which will be given in Section 5.3. Here, to derive the results, we crucially use the various contraction properties of the Markovian semi-groups, the generator of which are uniformly elliptic operators. The above mentioned various contraction properties include the ultracontractivity, through which, generally, we can derive the regularity of the Green kernel and then see the path property of the Markov process corresponding to the Markovian semi-group. Hence, as have been done in this thesis, by showing that the mild solution of the Keller-Segel systems are expressed by the
Markovian semi-groups, in particular possessing the property of ultracontractivity, we are in the situation where the biological phenomenon (activities) are understood as the dynamics of some stochastic processes. In Theorem 5.2 and 5.3, the existence and uniqueness of time local mild solution $(u, v)$ of (GKS) are shown. To construct the mild solution $(u, v)$ of (GKS), we pass through a standard argument of successive approximations by means of strongly continuous semi-groups. (cf., e.g. Kozono and Sugiyama [17]). We remark that Aida, Efendiev and Yagi [1] give a general framework several quasilinear equations corresponding to the Keller-Segel system. Some equations introduced in [7] and [14], may be discussed through the general framework given by [1]. There is a possibility that the general theorem given by [1] can be applied to the present problem by passing through careful considerations on the assumptions of our model (cf. [25]). Nevertheless, here we treat our problem through the direct iteration procedure mentioned above, the method possesses its own importance (cf. the final paragraph of this section). There still remain some problems on (GKS): Whether the solution of (GKS) blows up or not? How about the regularity of the mild solution? Considerations of these problems are postponed here, and they will be discussed in forthcoming papers. The author is also interested in considering the higher-dimensional case or the system of parabolic-parabolic type. This chapter is based on Yahagi [37].

Finally in Appendixes A and B, we give the proofs of important equations given in Chapter 3, and main propositions given in Chapter 4, respectively.

Acknowledgment

The author expresses her deep gratitude to Prof. Hiroaki Kikuchi for the helpful discussions and warm encouragement. Prof. Minoru.W.Yoshida is also deeply acknowledged for the discussions on the framework of the thesis.
2 The Keller-Segel system and the Brownian motion process

2.1 The Keller-Segel system as the biological model

We describe the life cycle of the cellular slime molds as follows. The cellular slime mold forms the structure like the plant called a fruit body finally. Then the spore released from a fruit body germinates, and it eats bacteria inhabit in the soil as feed, and increases in the state of the amoeba. After it eats whole of feed of bacteria in the surrounding area, it falls into starvation. Then it begins to release a chemical substance, that is cAMP, which attracts other cells. Hence they are gathering. Then a cell body moves to the lightning place, and it grows to a fruit body (see Figure 1). The Keller-Segel system is the biological model which describes the movement until a cellular slime mold falls in the hunger state and forms an aggregate.

2.2 The numerical computation on the one-dimensional Keller-Segel system

Here, we focus on the following one-dimensional Keller-Segel system with the Neumann boundary conditions: The Neumann boundary conditions are also referred to as reflection boundary conditions, and they mean that there is no out of cells and
chemotactic substance through the boundary \( \partial I = \{a, b\} \).

\[
\begin{align*}
\text{(KS)} \quad &\begin{cases}
  u_t = u_{xx} - \chi(uv_x)_x, & (x,t) \in I \times (0, \infty), \\
  v_t = v_{xx} - \gamma v + \alpha u, & (x,t) \in I \times (0, \infty), \\
  u_x(a,t) = u_x(b,t) = v_x(a,t) = v_x(b,t) = 0, & t \in (0, \infty), \\
  u(x,0) = \overline{u}(x), v(x,0) = \overline{v}(x), & x \in I,
\end{cases}
\end{align*}
\]

where \( I = (a,b) \) with some given \( a \) and \( b \) such that \(-\infty < a < b < \infty\) is a bounded open interval, and \( \chi, \alpha, \gamma \) are some given positive constants. The solutions \( u = u(x,t) \) and \( v = v(x,t) \) represent the cell density of the cellular slime molds and the cell concentration of the chemical substance that released by the cellular slime molds at the position \( x \), and at time \( t \), respectively.

We introduce the following example.

**Figure 2** Result of the numerical computation of Example 2.1
The figure is the graph of \( u = u(x,t) \) of Example 2.1.

**Example 2.1** Let parameters \( \chi, \alpha, \gamma, a, b \) and initial functions \( \overline{u}, \overline{v} \) in (KS) be as follows: \( \chi = 3, \alpha = \gamma = 1, a = -10, b = 10, \)

\[
\overline{u}(x) = \begin{cases}
  \cos(x + \pi) + 1 & (-2\pi \leq x \leq 2\pi), \\
  0 & (-10 < x < -2\pi, 2\pi < x < 10),
\end{cases}
\]

\[
\overline{v}(x) = \begin{cases}
  \cos x + 1 & (-\pi \leq x \leq \pi), \\
  0 & (-10 < x < -\pi, \pi < x < 10).
\end{cases}
\]

Then we have the above graph of \( u(x,t) \) by a direct, numerical computation. If we interpret Figure 2 as the biological model, it shows that two groups of the cells form an aggregate as time passes.
2.3 Brownian motion and heat equation

The one-dimensional heat equation is given by

\[ u_t = ku_{xx}, \]

with \( u = u(x,t) \) and a positive constant \( k \). Example 2.2, given below, shows a correspondence between the heat equation and the standard Brownian motion.

Example 2.2 We consider the following example with the initial condition of the heat equation:

\[
\begin{align*}
(H) \quad & \begin{cases} 
  u_t = \frac{1}{2} u_{xx}, & (x,t) \in \mathbb{R} \times (0, \infty), \quad (2.1) \\
  u(x,0) = \varpi(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
\]

where \( u = u(x,t) \) is the temperature of the object in the location \( x \) and at time \( t \). The fundamental solution \( K(x,t) \) of (2.1) is given by

\[ K(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(4t)}. \quad (3) \]

The solution of the initial problem of the heat equation (H) is given by the convolution of \( K(x,t) \) and the initial function \( \varpi \) as follows:

\[ u(x,t) = \int_{-\infty}^{\infty} K(x-y,t) \varpi(y) \, dy. \quad (4) \]

In fact, the relation \( K_t = \frac{1}{2} K_{xx} \) can be certified through a direct calculation with (3). By this and by performing differentiations for (4) (noting that for \( t > 0 \) the kernel \( K \) is smooth with respect to both variables), we obviously see that \( u(x,t) \) satisfies (H). On the other hand, we define a sequence of random variables \( \{X_n\} \) as follows: Suppose that we throw one coin repeatedly. At the \( k \)-th trial, if the coin is head then we define the random variable \( X_k = 1 \), and if it is tail then we set \( X_k = -1 \). Let \( S_n \) and \( B_n(x,t) \) be as follows:

\[ S_n = \sum_{k=1}^{n} X_k, \quad B_n(x,t) = \frac{S_{[nt]}}{\sqrt{n}} + x. \]

Then from the central limit theorem, \( B_n(x,t) \) converges to a certain stochastic process \( B(x,t) \) that follows the normal distribution with mean \( x \) and variance \( t \). That is, there exists a Brownian motion \( B(x,t) \) which holds the following equation:

\[ P \left( a \leq B(x,t) \leq b \right) = \int_a^b K(x-y,t) \, dy, \]

where \( B(x,0) = x \), and \( P \) is a probability measure on a measurable space of continuous path \( C([0,\infty); \mathbb{R}) \). Let \( E \) denote the expectation by means of the probabilistic
measure $P$. In the sequel, we adopt the similar notations (cf. Proposition 4.1). Then $u(x, t)$ given by (4) can be expressed the following equation:

$$u(x, t) = E[\pi(B(x, t)) \mid B(x, 0) = x].$$

The standard Brownian motion is defined as a Markov process from the mathematical viewpoint in the probability theory. The standard Brownian motion is a continuous process, and $B(0, 1)$ has the normal distribution with mean 0 and variance 1. In this thesis, we define stochastic processes $X(s)$ and $Y(s)$ which are independent standard Brownian motions each other and also give probabilistic expressions to the solution $(u, v)$ of (KS).
3 A probabilistic consideration on the Keller-Segel system

3.1 The existence and uniqueness of the solution of the Keller Segel system

It is known that (KS) has an unique time global classical solution \((u, v)\) under suitable conditions. (See Theorem 4.2 and Section 7 in [24].)

**Proposition 3.1** (Osaki and Yagi [24])

Suppose that the initial functions \(u, v\) satisfy the following conditions,

\[
\inf_{x \in I} u > 0, \quad \inf_{x \in I} v > 0, \quad u \in H_N^1(I), \quad v \in H_N^3(I).
\]

Then, there exists a unique time global solution \((u, v)\) of (KS) such that

\[
\begin{align*}
  u & \in C^1([0, \infty); L^2(I)) \cap C^1((0, \infty); H_N^2(I)) \cap C((0, \infty); H_N^3(I)), \\
v & \in C^1([0, \infty); H_N^1(I)) \cap C((0, \infty); H_N^3(I)) \cap C^1((0, \infty); H_N^4(I)),
\end{align*}
\]

where

\[
H_N^k(I) := \{ u \in H^k(I) \mid \frac{du}{dx}(a) = \frac{du}{dx}(b) = 0 \} \quad (k = 2, 3),
\]

\[
H_N^4(I) := \{ u \in H^4(I) \mid \frac{du}{dx}(a) = \frac{du}{dx}(b) = 0, \frac{d^3u}{dx^3}(a) = \frac{d^3u}{dx^3}(b) = 0 \}.
\]

Moreover, by Sobolev’s embedded theorem, from (5) and (6) we have

\[
\begin{align*}
  u & \in C^1([0, \infty); L^2(I)) \cap C((0, \infty); C_N^2(T)), \\
v & \in C^1([0, \infty); C_N(T)) \cap C((0, \infty); C_N^2(T)),
\end{align*}
\]

where

\[
C_N^2(T) := \{ u \in C^2(T) \mid \frac{du}{dx}(a) = \frac{du}{dx}(b) = 0 \}.
\]

In this chapter, our object is to give a probabilistic expression to the time global solution \((u, v)\) of (KS), and to consider the properties of the solution. To do so, before proceeding to the main section of the present chapter, we recall the fundamental formula in stochastic analysis.

3.2 Itô’s formula

As we see in Example 2, the solution of the initial problem of the heat equation is expressed by means of the expectation with the standard Brownian motion. K. Itô led to the following famous Itô’s formula to correspondence with stochastic differential equations and diffusion equations.
**Proposition 3.2** (Theorem 7.4, Bensoussan and Lions [6])

Let $u \in C^2(\mathbb{R})$. For any $x \in \mathbb{R}, t \in [0, \infty)$, let $X(s), t \leq s < \infty$, be the stochastic process defined by the following stochastic differential equation:

\[
\begin{align*}
  dX(s) &= b(X(s))ds + \sigma(X(s)) dB_s, \\
  X(t) &= x,
\end{align*}
\]

where $B_s$ is the standard Brownian motion defined on a complete probability space $(\Omega, F, P; F_t)$, with a filtration $(F_t)_{t \geq 0}$ and $b \in C^{1,1}(\mathbb{R} \times [0, \infty)), \sigma \in C^1(\mathbb{R})$. In addition, if we assume that there exists a positive constant $M$ such that $\sigma(y) \geq M$ for any $y \in \mathbb{R}$, then the following Itô's formula holds:

\[
\begin{align*}
  d\left(\frac{u(X(t))}{X(t)}\right) &= u'(X(t))\frac{b(X(t))}{X(t)} dt \\
  &\quad + \sigma(X(t)) dB(t) + \frac{1}{2} \sigma^2(X(t)) \cdot u''(X(t)) dt, \\
  u(X(t)) &= u(x).
\end{align*}
\]

The above Itô's formula is extended to general semi-martingale (see Section II-4 of Ikeda and Watanabe [15]), known as generalized Itô's formula. The first and the second equations (1.1) and (1.2) of (KS) are diffusion equations. We give the probabilistic expressions to the solution of backward equations of (KS) by using stochastic differential equations.

### 3.3 Main result

As has been seen in Proposition 3.1, (KS) has a unique time global classical solution $(u, v)$ under the initial conditions given in the same proposition. In our main theorem, Theorem 3.4, we give bounds for the solution $(u, v)$ of (KS). We prepare the backward equations of the Keller Segel system (KS) which is replaced by $T - t$ for any $T > 0$.

\[
\begin{align*}
  -\ddot{u}_t &= \ddot{u}_{xx} - \chi(\ddot{u}\dot{v}_x)_x & (x, t) &\in I \times (0, T), \\
  -\ddot{v}_t &= \ddot{v}_{xx} - \gamma\ddot{v} + \alpha \ddot{u} & (x, t) &\in I \times (0, T), \\
  \ddot{u}_x(a, t) &= \ddot{u}_x(b, t) = \ddot{v}_x(a, t) = \ddot{v}_x(b, t) = 0 & t &\in (0, T), \\
  \ddot{u}(x, T) &= \ddot{u}(x), \quad \ddot{v}(x, T) = \ddot{v}(x) & x &\in I.
\end{align*}
\]

Note that there exists a unique solution $(\ddot{u}, \ddot{v})$ of $(KS)^*$ that possesses sufficient regularities by using Proposition 3.1. For the solution $\ddot{u}(x, t)$ of the backward equations that has sufficient regularity (cf. (7), (8)), we can apply the Itô’s formula to the stochastic process $\{\ddot{u}(X(t), t)\}_{t \geq 0}$ composed by $\ddot{u}$ with some Itô process $X(t)_{t \geq 0}$, and can clearly derive an expression of $\ddot{u}$ by means of an expectation of a process through the standard discussion given e.g. Chapter VIII of [1]. We can derive the following results for the solution $(\ddot{u}, \ddot{v})$ of $(KS)^*$ (see Pardoux and Pengor [26], precisely see (14) below and its proof). We can also treat a solution of the forward
equations (KS) and discuss a probabilistic expression of it (as was seen in Example 2), but in order to do so, we have to pass through another careful discussion on interchanging of semi-group and its generator corresponding to the Itô process, namely, we need to pass through a discussion on identifications between a solution of a stochastic differential equation and a diffusion process defined through Markov semi-group see e.g., Chapter IV of [15] (see also Ma and Röckner [19]).

**Proposition 3.3** Suppose that the conditions given by Proposition 3.1 are satisfied. Let \((\hat{u}, \hat{v})\) be a classical solution of \((KS)^*\) in \(I \times (0, T)\), and let the stochastic processes \(X(s), Y(s)\) be the solutions of the following stochastic differential equations (10), (11) respectively:

\[
\begin{align*}
\begin{cases}
\frac{dX(s)}{ds} = -\chi \hat{v}_x(X(s), s) 1_I(X(s)) \ ds + \sqrt{2} 1_I(X(s)) \ dB_s + d\phi_1(s), \\
X(t) = x.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\frac{dY(s)}{ds} = \sqrt{2} 1_I(Y(s)) \ dB_s + d\phi_2(s), \\
Y(t) = x,
\end{cases}
\end{align*}
\]

where \(1_I(\cdot)\) is the indicator function, \(1_I(z) = \begin{cases} 1 & (z \in I) \\ 0 & \text{(otherwise)} \end{cases}\), \(\phi_1(s)\) and \(\phi_2(s)\) are the local time of the process \(\{X(s)\}\) and \(\{Y(s)\}\) respectively by which the boundary points \(a\) and \(b\) become reflection boundaries (see Section IV of [15], and Yoshida [39] and references therein). The conditions \(X(t) = x\) and \(Y(t) = x\) imply that the position of stochastic processes \(X(s)\) and \(Y(s)\) at time \(t\) are \(x\). Then we have the following probabilistic expressions (i.e. the expressions by means of Markovian semi-groups):

\[
\hat{u}(x, t) = E [\pi(X(T)) e^{-\int^T_t \chi \hat{v}_xx(X(\tau), \tau) \ d\tau} \ | \ X(t) = x].
\]

\[
\hat{v}(x, t) = E [\pi(Y(T)) e^{-\gamma(T-t)} \ | \ Y(t) = x] + E [\int^T_t \alpha \hat{u}(Y(s), s) e^{-\gamma(s-t)} \ ds \ | \ Y(t) = x],
\]

where \(B_s\) is the standard Brownian motion, and \(E\) is the expectation depended on expectation measure \(P\).

**Proof of Proposition 3.3.**
We consider the following functional with stochastic process \(X(s)\)

\[
\hat{u}(X(s), s) e^{-\int^s_t \chi \hat{v}_xx(X(\tau), \tau) \ d\tau} (s \geq t).
\]

In particular, by Proposition 3.2, the generalized Itô’s formula and (9.1), we have

\[
d\left(\hat{u}(X(s), s) e^{-\int^s_t \chi \hat{v}_xx(X(\tau), \tau) \ d\tau}\right) = \sqrt{2} \hat{u}_x(X(s), s) \ dB_s e^{-\int^s_t \chi \hat{v}_xx(X(\tau), \tau) \ d\tau}.
\]

(14)
Refer to Appendix A.1 below for more detailed proof.

We integrate both sides of (14) from \( t \) to \( T \), and take the expectation \( E \), then we have the following equation:

\[
E[\tilde{u}(X(T), T)e^{-\int_t^T \chi \tilde{v}_{xx}(X(\tau), \tau) \, d\tau} - \tilde{u}(X(t), t) \mid X(t) = x] = E[\sqrt{2} \int_t^T e^{-\int_t^s \chi \tilde{v}_{xx}(X(\tau), \tau) \, d\tau} \tilde{u}_x(X(s), s) \, dB_s \mid X(t) = x].
\]  

(15)

Since the right-hand side of (15) equals to zero and \( \tilde{u}(x, T) \) equals to \( u(x) \), we have the equation (12). By considering the functional \( \tilde{v}(Y(s), s)e^{-\gamma \int_t^s \, d\tau} \), we obtain (13) in a similar fashion (Refer to Appendix A.2 below for the proof).

By taking \( s \) as the time to go, for the backward equations (KS)*, from (12) we have

\[
\tilde{u}(x, T - s) = E[\overline{u}(X(T))e^{-\int_T^{T-s} \chi \overline{v}_{xx}(X(\tau), \tau) \, d\tau} \mid X(T - s) = x].
\]  

(16)

Noting the regularity given by Proposition 4.1, for each \( T > 0 \), by setting

\[
m := \inf_{(x,t)\in I\times[0,T)} v_{xx}(x,t), \quad M := \sup_{(x,t)\in I\times[0,T)} v_{xx}(x,t),
\]

from (16), we immediately have

\[
E[\overline{u}(X(T)) \mid X(T - s) = x] \cdot e^{-s M} \leq \tilde{u}(x, T - s) \leq E[\overline{u}(X(T)) \mid X(T - s) = x] \cdot e^{-s m}.
\]

Therefore, we have the following inequality:

\[
\inf_{x \in I} \pi(x) \cdot e^{-s M} \leq \tilde{u}(x, T - s) \leq \sup_{x \in I} \pi(x) \cdot e^{-s m}.
\]  

(17)

Note that because of the Neumann boundary condition for \( v \), \( M > 0 \) and \( m < 0 \) hold.

From (13), for the solution \( \tilde{v} \) of the backward equation system (KS)*, we have

\[
\tilde{v}(x, T - s) = E[\overline{v}(Y(T))e^{-\gamma s} \mid Y(T - s) = x]
\]

\[
+ E[\int_{T-s}^T \alpha \tilde{u}(Y(\tau), \tau)e^{-\gamma (\tau-T+s)} \, d\tau \mid Y(T - s) = x] =: I + II.
\]

By (11), since the process \( Y \) is a Brownian motion with reflection boundaries, we have the following equation:

\[
I = e^{-\gamma s} E[\overline{v}(Y(T)) \mid Y(T - s) = x] = e^{-\gamma s}\left(\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{a-b}\right)^2s} \cos\left(\frac{n\pi}{b-a}(x-a)\right)\right),
\]

12
where \( A_n = \frac{2}{b - a} \int_a^b \pi(x) \cos \frac{n \pi}{b - a} (x - a) \, dx \) \((n = 0, 1, 2, \ldots)\).

For the integrand of \( II \), from (17) it holds that

\[
\tilde{u}(Y(\tau), \tau) = \tilde{u}(Y(\tau), T - (T - \tau)) \leq \sup_{x \in I} \pi \cdot e^{-(T-\tau)\chi m}.
\]

Thus we have

\[
II = E \left[ \int_{T-s}^{T} \alpha \, \tilde{u}(Y(\tau), \tau) e^{-\gamma(\tau - T + s)} \, d\tau \mid Y(T - s) = x \right]
\]

\[
\leq \alpha \int_{T-s}^{T} \sup_{x \in I} \pi \cdot e^{-(T-\tau)\chi m} e^{-\gamma(\tau - T + s)} \, d\tau
\]

\[
= \alpha \sup_{x \in I} \pi \cdot e^{-T\chi m + \gamma T - \gamma s} \int_{T-s}^{T} e^{(\chi m - \gamma)\tau} \, d\tau
\]

\[
= \alpha \sup_{x \in I} \pi \cdot e^{-T\chi m + \gamma T - \gamma s} \frac{1}{\chi m - \gamma} \{ e^{(\chi m - \gamma)T} - e^{(\chi m - \gamma)(T-s)} \}
\]

\[
= \alpha \sup_{x \in I} \pi \cdot e^{-T\chi m + \gamma T - \gamma s} \frac{1}{\chi m - \gamma} e^{(\chi m - \gamma)T} (1 - e^{(\chi m - \gamma)(-s)})
\]

\[
= \alpha \sup_{x \in I} \pi \cdot e^{-\gamma s} \frac{1}{\chi m - \gamma} (1 - e^{(\gamma M)(-s)})
\]

\[
= \alpha \sup_{x \in I} \pi \cdot \frac{e^{-\gamma M s} - e^{-\gamma s}}{\gamma - \chi m} \quad (\gamma - \chi M > 0). \tag{18}
\]

Similarly for the estimate of lower bound, we have

\[
II \geq \left\{ \begin{array}{ll}
\alpha \inf_{x \in I} \pi \cdot \frac{e^{-\gamma M s} - e^{-\gamma s}}{\gamma - \chi M} & (\gamma - \chi M \neq 0), \\
\alpha \inf_{x \in I} \pi \cdot se^{\gamma s} & (\gamma - \chi M = 0).
\end{array} \right.
\]

Refer to Appendix 6.3 for the proof.

As a consequence, since \( u(x, t) = \tilde{u}(x, T-t) \), \( v(x, t) = \tilde{v}(x, T-t) \), we obtain the following main theorem which gives bounds for the solution of (KS).

**Theorem 3.4** Suppose that the conditions given in Proposition 3.1 are satisfied. Let \((u, v)\) be the classical solution of (KS) defined through the solution \((\tilde{u}, \tilde{v})\) of (KS)* by the change of variable \( t \) by \( T-t \). Then, for any \( s \) \((0 < s < T)\), the following inequalities hold:

\[
\inf_{x \in I} \pi(x) \cdot e^{-\gamma M s} \leq u(x, s) \leq \sup_{x \in I} \pi(x) \cdot e^{-\gamma M s}, \tag{19}
\]

\[
K_M(s) \leq v(x, s) - \Phi(\pi)(x, s) \leq K_m(s), \tag{20}
\]

13
where
\[
\Phi(\overline{v})(x, s) := e^{-\gamma s} \left( \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{\pi n}{b-a}\right)^2s} \cos \frac{n\pi}{b-a}(x-a) \right),
\]
\[
A_n := \frac{2}{b-a} \int_a^b \overline{v}(x) \cos \frac{n\pi}{b-a}(x-a) \, dx \quad (n = 0, 1, 2, \cdots),
\]
\[
K_m(s) := \alpha \sup_{x \in I} e^{\frac{-\chi ms - e^{-\gamma s}}{\gamma - \chi m}},
\]
\[
K_M(s) := \begin{cases} 
\alpha \inf_{x \in I} e^{\frac{-\chi Ms - e^{-\gamma s}}{\gamma - \chi M}} & (\gamma - \chi M \neq 0), \\
\alpha \inf_{x \in I} e^{\gamma s} & (\gamma - \chi M = 0)
\end{cases}.
\]

Remark 3.5 Theorem 3.4 would be an application or modification of maximal principle in usual analysis. However it can be proved easily through the stochastic analytic methods, as we have seen above.

Remark 3.6 The above inequalities (19) and (20) give the good estimates of \(u(x, t)\) and \(v(x, t)\) for small \(t > 0\), respectively. Indeed, for small \(t > 0\) the solution \((u, v)\) is so similar to \((\overline{u}, \overline{v})\) as we know the following example.

![Figure 3](image-url)  
**Figure 3** Result of the numerical computation of Example 3.7 \((T = 1)\)  
The figure of the left-hand side is the graph of \(u = u(x, t)\), and the righthand side is the graph of \(v = v(x, t)\).

Example 3.7 Let parameters \(\chi, \alpha, \gamma, a, b\) and initial functions \(\overline{u}, \overline{v}\) in \((KS)\) and the time \(T\) in \((KS)^*\) be as follows: \(\chi = 2, \alpha = 1, \gamma = 2, a = -10, b = 10, T = 1,\)
\[
\overline{u}(x) = \begin{cases} 
2 + \cos x & (-\pi \leq x \leq \pi), \\
1 & (-10 < x < -\pi, \pi < x < 10),
\end{cases} \quad \overline{v}(x) \equiv 1.
\]
Then we have Figure 3 by a direct numerical computation. In case where $T$ is not so large, we see that for $0 \leq t \leq T$ the solution $(u, v)$ is so similar to $(\bar{u}, \bar{v})$.

![Figure 3](image1.png)

Figure 4  Result of the numerical computation of Example 3.8 ($T = 20$)
The figure of the left-hand side is the graph of $u = u(x, t)$, and the righthand side is the graph of $v = v(x, t)$.

Example 3.8 Let parameters $\chi, \alpha, \gamma, a, b$ and initial functions $\bar{u}, \bar{v}$ in (KS) be the same as Example 3.7, but in the present case take $T = 20$. We have the above graphs of $u(x, t)$ and $v(x, t)$ by a direct numerical computation. Left figure is the graph of $u = u(x, t)$ and right one is the graph of $v = v(x, t)$. In case where $T$ is large, we see that for $0 \leq t \leq T$ the solution $(u, v)$ becomes different from $(\bar{u}, \bar{v})$.

![Figure 4](image2.png)
4 Asymptotic behavior of solutions to the one-dimensional Keller-Segel system with small chemotaxis

Throughout this chapter, we denote by $L^r \equiv L^r(I)$, the usual Lebesgue space on $I$ with the norm $\|u\|_{L^r} \equiv (\int_I |u(x)|^r \, dx)^{1/r}$ for $1 \leq r < \infty$, and $\|u\|_{L^\infty} \equiv \text{ess sup}_{x \in I} |u(x)|$. The Sobolev space $W^{m,r}(I)$, $m = 1, 2, \ldots, 1 < r < \infty$ is the space of all functions $u$ on $I$ such that $\|u\|_{W^{m,r}} \equiv (\sum_{i=1}^m \|D^i u\|_{L^r})^{1/r} < \infty$, with the derivative $D$ with respect to the variable $x$.

4.1 Existing results

In this section, we give some propositions that will play an important role in our main results given in the next section. It is known that (KS) defined by (1.1) – (1.3) has a unique global-in-time classical solution $(u, v)$ under suitable initial conditions (cf. Section 7 in Osaki and Yagi [24]):

**Proposition 4.1** (Osaki and Yagi [24]) Suppose that the initial data $\bar{u}, \bar{v}$ satisfy the following conditions:

$$\bar{u}, \bar{v} \in W^{1,2}(I), \inf_{x \in I} \bar{u} > 0, \inf_{x \in I} \bar{v} > 0.$$ 

Then, there exists a unique global-in-time classical solution $(u, v)$ of (KS).

The following Proposition 4.2 is derived easily by using integration by parts and the Neumann boundary conditions.

**Proposition 4.2** Assume that $(u, v)$ is the solution of (KS). Then the following identity holds:

$$\int_I u(x,t) \, dx = \int_I \bar{u}(x) \, dx. \quad (21)$$

The identity (21) is called as “mass conservation law”, and it tells us that the amounts of the cellular slime molds do not change in time. In the probability theory, functions that possess the mass conservation law can be interpreted as probability densities (cf. Yahagi [35]).

Now, we consider the following non-homogeneous heat equation (22.1) with the Neumann boundary conditions (22.2).

$$(\text{Heat}) \left\{ \begin{array}{ll} w_t = w_{xx} + z & \text{in} \ I \times (0, \infty), \\ w_x(a, t) = w_x(b, t) = 0 & \text{in} \ (0, \infty), \\ w(x, 0) = \bar{w}(x) & \text{in} \ I, \end{array} \right. \quad (22.1)$$

where $I = (a, b)$, $\bar{w} \in L^2(I)$ and $z \in C([0, \infty); L^2(I))$. 

16
Proposition 4.3 The solution \( w \) of (Heat) is given by the following formula:

\[
w(x, t) = \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} \left( T_n(0) + \int_0^t z_n(\tau) e^{\lambda_n^2 \tau} d\tau \right) \cos \lambda_n(x - a),
\]

where

\[
\lambda_n = \frac{\pi n}{b - a} \quad (n \geq 0),
\]

\[
T_n(0) = \frac{2}{b - a} \int_a^b \overline{w}(x) \cos \lambda_n(x - a) \, dx \quad (n \geq 1),
\]

\[
T_0(0) = \frac{1}{b - a} \int_a^b \overline{w}(x) \, dx,
\]

\[
z_n(t) = \frac{2}{b - a} \int_a^b z(x, t) \cos \lambda_n(x - a) \, dx \quad (n \geq 1),
\]

\[
z_0(t) = \frac{1}{b - a} \int_a^b z(x, t) \, dx.
\]

We will give the proof of Proposition 4.3 in Appendix B.1 below.

To prove our main theorems given by Section 4.2, we efficiently use the following two propositions, known as the Poincaré inequality (see e.g. Section 7 in Gilbarg and Trudinger [12], Chapter 3 in Mizohata [22]), and the Gronwall inequality (see e.g. Appendix B in Evans [10], respectively).

Proposition 4.4 Let \( I = (a, b) \) be the given open interval.
There exists a positive constant \( C = C(a, b) \) such that for \( u \in W^{1,2}(I) \) the following inequality holds:

\[
\|u - M_u\|_{L^2} \leq C \|Du\|_{L^2},
\]

where \( M_u := \frac{1}{b - a} \int_I u(y) \, dy \).

Proposition 4.5 Let \( J = (t_0, t_1) \) be the open interval. Suppose that \( u \in C^1(J) \) and \( \beta \in C(J) \) that satisfy

\[
\frac{du}{dt} \leq \beta(t) \, u(t), \quad (t \in J).
\]

Then, the following inequality holds:

\[
u(t) \leq u(c) \exp\left(\int_c^t \beta(s) \, ds\right),
\]

where \( t_0 < c \leq t < t_1 \).

Finally in this section, we prepare the following proposition which is an application of Proposition 4.5.
**Proposition 4.6** Let $F \in C^1(0, \infty)$ be a nonnegative function, and let $G \in C(0, \infty)$ be a nonnegative function such that $\lim_{t \to \infty} G(t) = 0$. Assume that the following differential inequality holds:

$$F'(t) \leq -kF(t) + lG(t),$$

where $k$ and $l$ are some given positive constants. Then, it holds that

$$\lim_{t \to \infty} F(t) = 0.$$

In Appendix B.2 below, we will give the proof of Proposition 4.6.

### 4.2 Main Results and their proofs

Recall that the second term on the right-hand side of (1.1) means that the cells move with the speed $\chi v_x$. Intuitively, it is expected that if $\chi v_x$ is small enough, then the diffusion phenomenon is stronger than the concentration phenomenon on (KS). As a result, it is expectable that the solution $u$ of (KS) will converge to a constant as the time goes by. In fact, we obtain the following main result.

**Theorem 4.7** Let $\overline{u}, \overline{v} \in W^{1,2}(I)$ and $\inf_{x \in I} \overline{u} > 0$, $\inf_{x \in I} \overline{v} > 0$, and let $(u, v)$ be the global-in-time classical solution of (KS). Suppose that one of the following two conditions (i), (ii) holds:

(i) Assume that there exists $t_* > 0$ such that for any $t \geq t_*$, the following inequality holds:

$$\chi C \|v_x(\cdot, t)\|_{L^\infty} < 1,$$

where $C = C(a, b)$ is a constant appearing in the Poincaré inequality (24). Furthermore, assume that

$$\lim_{t \to \infty} \|v_{xx}(\cdot, t)\|_{L^2} = 0.$$

(ii) Assume that

$$\lim_{t \to \infty} \|v_x(\cdot, t)\|_{L^\infty} = 0.$$

Then, it holds that

$$\lim_{t \to \infty} \|u(\cdot, t) - M\|_{L^2} = 0,$$

$$\lim_{t \to \infty} \|v(\cdot, t) - \frac{\alpha M}{\gamma}\|_{L^2} = 0,$$

where $M := \frac{1}{b - a} \int_a^b \overline{u}(x) \, dx$.

Note that by the “mass conservation law” (21), if the solution $u$ of (KS) converges to a constant $C$, then it must hold that $C = \frac{1}{b - a} \int_a^b \overline{u}(x) \, dx$. Before we prove this main theorem, we prepare the following system (KS*) which is obtained by
substituting $\chi = 0$ in (KS).

\[
\begin{cases}
\hat{u}_t = \hat{u}_{xx} & \text{in } I \times (0, \infty), \\
\tilde{v}_t = \tilde{v}_{xx} - \gamma \tilde{v} + \alpha \hat{u} & \text{in } I \times (0, \infty), \\
\hat{u}(a, t) = \hat{u}(b, t) = \tilde{v}(a, t) = \tilde{v}(b, t) = 0 & \text{in } (0, \infty), \\
\hat{u}(x, 0) = \overline{u}(x), \tilde{v}(x, 0) = \overline{v}(x) & \text{in } I. 
\end{cases}
\tag{29.1}
\]

Note that (KS') is linear, while (KS) is nonlinear. At first, we discuss the asymptotic behavior of $(\hat{u}, \tilde{v})$, which is the solution of (KS'), and we proceed to the discussion about (KS). Fortunately, it is possible to solve (KS'). Indeed, we have the following proposition.

**Proposition 4.8** Let $\overline{u}, \overline{v} \in W^{1,2}(I)$ and $\inf_{x \in I} \overline{u} > 0, \inf_{x \in I} \overline{v} > 0$. Then the classical solution $(\hat{u}, \tilde{v})$ of (KS) is given as follows:

\[
\hat{u}(x, t) = M + \sum_{n=1}^{\infty} A_n \cos \lambda_n(x-a) e^{-\lambda_n^2 t},
\tag{30}
\]

\[
\tilde{v}(x, t) = e^{-\gamma t} \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} \left( T_n(0) + \alpha \int_0^t \hat{u}(\tau) e^{(\gamma + \lambda_n^2) \tau} d\tau \right) \cos \lambda_n(x-a),
\tag{31}
\]

where $\lambda_n$ is the number given in Proposition 4.3, and

\[
A_n = \frac{2}{b-a} \int_a^b \overline{u}(x) \cos \lambda_n(x-a) \, dx \, (n \geq 1),
\]

\[
T_n(0) = \frac{2}{b-a} \int_a^b \overline{v}(x) \cos \lambda_n(x-a) \, dx \, (n \geq 1),
\tag{32}
\]

\[
T_0(0) = \frac{1}{b-a} \int_a^b \overline{v}(x) \, dx =: N,
\tag{33}
\]

\[
\hat{u}_n(t) = \frac{2}{b-a} \int_a^b \hat{u}(x, t) \cos \lambda_n(x-a) \, dx \, (n \geq 1),
\tag{34}
\]

\[
\hat{u}_0(t) = \frac{1}{b-a} \int_a^b \hat{u}(x, t) \, dx = \frac{1}{b-a} \int_a^b \overline{u}(x) \, dx = M.
\]

By using Proposition 4.8, we have the following theorem.

**Theorem 4.9** For the solution $(\hat{u}, \tilde{v})$ of (KS) given by (30) and (31), the following two hold:

\[
\lim_{t \to \infty} \| \hat{u}(\cdot, t) - M \|_{L^\infty} = 0,
\tag{35}
\]

\[
\lim_{t \to \infty} \| \tilde{v}(\cdot, t) - \frac{\alpha M}{\gamma} \|_{L^\infty} = 0.
\tag{36}
\]
(Proof of Proposition 4.8.) It is easy to solve (29.1) with (29.3) and (29.4) (see e.g. Section 12.3 in Kreyszig [18]). Here, we give the proof of (31) only. By transformation of variable \( \tilde{v}(x, t) = e^{-\gamma t} \tilde{w}(x, t) \), it holds that

\[
\tilde{v}_t = -\gamma e^{-\gamma t} \tilde{w} + e^{-\gamma t} \tilde{w}_t, \quad \tilde{v}_x = e^{-\gamma t} \tilde{w}_x, \quad \tilde{v}_{xx} = e^{-\gamma t} \tilde{w}_{xx}.
\]

Therefore we have the following differential equation instead of (10.2),

\[
\tilde{w}_t = \tilde{w}_{xx} + \alpha \tilde{v} e^{\gamma t}.
\]

Also, we have

\[
\tilde{w}(x, 0) = \tilde{v}(x).
\]

Then, by substituting \( w = \tilde{w}, z = \alpha \tilde{v} e^{\gamma t} \) in Proposition 4.3, it follows that

\[
\tilde{w}(x, t) = \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} \left( T_n(0) + \int_0^t \tilde{u}_n(\tau) e^{\lambda_n^2 \tau} d\tau \right) \cos \lambda_n(x - a),
\]

where \( T_n(0) \) and \( u_n(t) \) are the same as (32), (33), (34) and \( z_n(t) = \alpha e^{\gamma t} \tilde{u}_n(t) \). Finally, we obtain the required formula:

\[
\tilde{v}(x, t) = e^{-\gamma t} \tilde{w}(x, t) = e^{-\gamma t} \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} \left( T_n(0) + \alpha \int_0^t \tilde{u}_n(\tau) e^{(\gamma + \lambda_n^2) \tau} d\tau \right) \cos \lambda_n(x - a).
\]

\[\blacksquare\]

(Proof of Theorem 4.9.) Since \( e^{-\lambda_n^2 t} = \frac{1}{e^{\lambda_n^2 t}} \leq \frac{1}{\lambda_n^2 t + 1} \leq \frac{1}{\lambda_n^2 t} \) for any \( \lambda_n^2 t > 0 \), it holds that

\[
\sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \leq \frac{1}{t} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{(b - a)^2}{t \pi^2} \cdot \frac{\pi^2}{6} = \frac{(b - a)^2}{6t}.
\]

Now, by using (37), we shall show the first requirement. By (30), we have

\[
|\tilde{u}(x, t) - M| = \left| \sum_{n=1}^{\infty} A_n \cos \lambda_n(x - a) e^{-\lambda_n^2 t} \right|
\leq \sum_{n=1}^{\infty} |A_n| e^{-\lambda_n^2 t} \\
\leq \frac{2M}{t} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{2M(b - a)^2}{t \pi^2} \cdot \frac{\pi^2}{6} = \frac{M(b - a)^2}{3t}.
\]

It follows that

\[
||\tilde{u}(\cdot, t) - M||_{L^\infty} \leq \frac{M(b - a)^2}{3t} \to 0 \ (t \to \infty).
\]

Then by (31), we have

\[
\tilde{v}(x, t) = e^{-\gamma t} \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} \left( T_n(0) + \alpha \int_0^t \tilde{u}_n(\tau) e^{(\gamma + \lambda_n^2) \tau} d\tau \right) \cos \lambda_n(x - a)
\]
\[ e^{-\gamma t} \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} T_n(0) \cos \lambda_n(x-a) + \alpha e^{-\gamma t} \int_0^t \hat{u}_0(\tau) e^{\gamma \tau} d\tau \\
\quad + e^{-\gamma t} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \alpha \int_0^t \hat{u}_n(\tau) e^{(\gamma+\lambda_n^2) \tau} d\tau \cos \lambda_n(x-a) \]
\[ =: I_1(x,t) + I_2(t) + I_3(x,t). \]

We see that
\[ |I_1(x,t)| \leq e^{-\gamma t} \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} |T_n(0)| \cos \lambda_n(x-a) \leq 2N e^{-\gamma t} \sum_{n=0}^{\infty} e^{-\lambda_n^2 t}. \]  \hspace{1cm} (38)

By using (37) again, it follows that
\[ \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \leq 1 + \frac{(b-a)^2}{6t} \quad (t > 0). \]  \hspace{1cm} (39)

From (38) and (39), we have
\[ \sup_I |I_1(x,t)| \leq 2N e^{-\gamma t} (1 + \frac{(b-a)^2}{6t}) \to 0 \quad (t \to \infty). \]

Next, we consider \( I_2(t) \).
\[ I_2(t) = \alpha e^{-\gamma t} \int_0^t \hat{u}_0(\tau) e^{\gamma \tau} d\tau = \alpha e^{-\gamma t} \int_0^t M e^{\gamma \tau} d\tau \]
\[ = \alpha e^{-\gamma t} M \cdot \frac{1}{\gamma} (e^{\gamma t} - 1) \to \frac{\alpha M}{\gamma} \quad (t \to \infty). \]

Finally let us evaluate \( I_3(x,t) \). From (30) and (34), for \( n \geq 1 \), we have
\[ \hat{u}_n(t) = \frac{2}{b-a} \int_a^b \hat{u}(x,t) \cos \lambda_n(x-a) \, dx \]
\[ = \frac{2}{b-a} \int_a^b \left( M + \sum_{m=1}^{\infty} A_m \cos \lambda_m(x-a) e^{-\lambda_m^2 t} \right) \cos \lambda_n(x-a) \, dx \]
\[ = \frac{2}{b-a} \sum_{m=1}^{\infty} A_m e^{-\lambda_m^2 t} \int_a^b \cos \lambda_m(x-a) \cos \lambda_n(x-a) \, dx \]
\[ = \frac{2}{b-a} A_n e^{-\lambda_n^2 t} \int_a^b \cos \lambda_n^2(x-a) \, dx \]
\[ = A_n e^{-\lambda_n^2 t}. \]

Therefore, it follows that
\[ I_3(x,t) = e^{-\gamma t} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \alpha \int_0^t \hat{u}_n(\tau) e^{(\gamma+\lambda_n^2) \tau} d\tau \cos \lambda_n(x-a) \]
\[ = e^{-\gamma t} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \alpha \int_0^t A_n e^{\gamma \tau} d\tau \cos \lambda_n(x-a) \]

(21)
\[
= \alpha e^{-\gamma t} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} A_n \cos \lambda_n(x-a) \int_0^t e^{\gamma \tau} \, d\tau
\]
\[
= \alpha e^{-\gamma t} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} A_n \cos \lambda_n(x-a) \frac{1}{\gamma} (e^{\gamma t} - 1)
\]
\[
= \frac{\alpha}{\gamma} \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} A_n \cos \lambda_n(x-a) (1 - e^{-\gamma t}) \rightarrow 0 \quad (t \to \infty).
\]

Hence, we obtain the following result:
\[
\|\hat{v}(\cdot, t) - \frac{\alpha M}{\gamma}\|_{L^\infty} \leq \|I_1(\cdot, t)\|_{L^\infty} + \|I_2(t) - \frac{\alpha M}{\gamma}\| + \|I_3(\cdot, t)\|_{L^\infty} \to 0 \quad (t \to \infty).
\]

Thus (36) has been proven. ■

At the end of this chapter, we give a proof of Theorem 4.7.

(Proof of Theorem 4.7.) First of all, we shall show (26) under the condition (i). From the first equation (1.1) of (KS), we see easily
\[
(u - M)_t = (u - M)_{xx} - \chi (uv_x)_x. \tag{40}
\]

Multiplying (40) by \(u - M\) and integrating the product on the interval \(I\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_I (u - M)^2 \, dx = \int_I (u - M) (u - M)_{xx} \, dx - \chi \int_I (u - M)(uv_x)_x \, dx. \tag{41}
\]

By the integration by parts formula, from (41) it follows that
\[
\frac{1}{2} \frac{d}{dt} \int_I (u - M)^2 \, dx = -\int_I (u - M)_x^2 \, dx - \chi \int_I (u - M)(uv_x)_x \, dx =: J_1(t) + J_2(t). \tag{42}
\]

We know that
\[
J_1(t) = -\int_I (u - M)_x^2 \, dx = \int_I (u_x)^2 \, dx = -\|u_x(\cdot, t)\|_{L^2}^2.
\]

Note that
\[
(uv_x)_x = u_xv_x + uv_{xx} = ((u - M)v_x)_x + Mv_{xx}.
\]

Then by the Hölder inequality, we see
\[
J_2(t) = -\chi \int_I (u - M)(uv_x)_x \, dx
\]
\[
= \chi \int_I (u - M)((u - M)v_x)_x + Mv_{xx} \, dx
\]
\[
= \chi \int_I (u - M)_x(u - M)v_x \, dx - \chi \int_I (u - M)Mv_{xx} \, dx
\]
\[
\leq \chi \|v_x(\cdot, t)\|_{L^\infty} \|u_x(\cdot, t)\|_{L^2} \|u(\cdot, t) - M\|_{L^2} + \chi M \|v_{xx}(\cdot, t)\|_{L^2} \|u(\cdot, t) - M\|_{L^2}.
\]

22
Suppose that there exists $t_*>0$ such that the following property holds (see (25)):

$$p := 1 - \chi C \|v_x(\cdot, t)\|_{L^\infty} > 0 \quad (t \geq t_*),$$

where $C = C(a, b)$ is a constant appearing in the Poincaré inequality (24). Then, we have

$$J_1(t) + J_2(t) \leq -p \|u_x(\cdot, t)\|_{L^2}^2 + \chi M \|v_{xx}(\cdot, t)\|_{L^2} \|u(\cdot, t) - M\|_{L^2} \leq -\frac{p}{C^2} \|u(\cdot, t) - M\|_{L^2}^2 + \chi M \|v_{xx}(\cdot, t)\|_{L^2} \|u(\cdot, t) - M\|_{L^2} \leq -\frac{p}{2C^2} \|u(\cdot, t) - M\|_{L^2}^2 + \frac{\chi^2 M^2}{4} \|v_{xx}(\cdot, t)\|_{L^2}^2.$$  

Therefore from (42) and above inequalities, it holds that

$$\frac{d}{dt} \|u(\cdot, t) - M\|_{L^2}^2 \leq -\frac{p}{C^2} \|u(\cdot, t) - M\|_{L^2}^2 + \frac{\chi^2 M^2}{4} \|v_{xx}(\cdot, t)\|_{L^2}^2. \quad (43)$$

From (43) and the assumption $\lim_{t \to \infty} \|v_{xx}(\cdot, t)\|_{L^2} = 0$, by applying Proposition 4.6, we find

$$\|u(\cdot, t) - M\|_{L^2}^2 \to 0 \quad (t \to \infty),$$

and thus we have proved (26).

Next, we shall show (26) under the condition (ii). As well as the above, multiplying the first equation (1.1) of (KS) by $u$ and integrating the product on the interval $I$, we have

$$\frac{1}{2} \frac{d}{dt} \int_I u^2 \, dx = \int_I u u_{xx} \, dx - \chi \int_I u(uv_x)_x \, dx.$$

By the integration by parts formula, it follows that

$$\frac{1}{2} \frac{d}{dt} \int_I u^2 \, dx = -\int_I (u_x)^2 \, dx + \chi \int_I u_x(uv_x) \, dx =: k_1(t) + k_2(t).$$

Because of the “mass conservation law” (21), it holds that $\frac{d}{dt} \int_I u \, dx = 0$. Thus we have

$$\frac{d}{dt} \|u(\cdot, t) - M\|_{L^2}^2 = \frac{d}{dt} \int_I (u^2 - 2Mu + M^2) \, dx = \frac{d}{dt} \int_I u^2 \, dx = \frac{d}{dt} \|u(\cdot, t)\|_{L^2}^2.$$  

Also it holds that

$$k_1(t) = -\int_I (u_x)^2 \, dx = -\|u_x(\cdot, t)\|_{L^2}^2.$$  

Then, by the Hölder inequality, we see

$$k_2(t) = \chi \int_I u_x(uv_x) \, dx \leq \chi \|v_x(\cdot, t)\|_{L^\infty} \int_I |u_xu| \, dx \leq \chi \|v_x(\cdot, t)\|_{L^\infty} \|u_x(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^2}. \quad (23)$$
It follows that
\[
\begin{align*}
    k_1(t) + k_2(t) &\leq -\|u_x(\cdot, t)\|_{L^2}^2 + \chi\|v_x(\cdot, t)\|_{L^\infty} \|u_x(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^2} \\
    &\leq -\frac{1}{2}\|u_x(\cdot, t)\|_{L^2}^2 + \chi\|v_x(\cdot, t)\|_{L^\infty} \|u_x(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^2} - \frac{1}{2}\|u_x(\cdot, t)\|_{L^2}^2 \\
    &\leq \frac{1}{2}\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2 \|u(\cdot, t)\|_{L^2} - \frac{1}{2}\|u_x(\cdot, t)\|_{L^2}^2 \\
    &\leq \frac{1}{2}\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2 \|u(\cdot, t)\|_{L^2} - M + M\|v_x(\cdot, t)\|_{L^2}^2 \\
    &\leq \frac{1}{2}\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2 \big(\|u(\cdot, t)\|_{L^2} - M\|v_x(\cdot, t)\|_{L^2}^2 + 2M\sqrt{b-a}\|u(\cdot, t) - M\|_{L^2}^2 \big) \\
    &\hspace{2em}+ M^2(b-a) - \frac{1}{2C^2} \|u(\cdot, t)\|_{L^2}^2 \\
    &\leq -(\frac{1}{4C^2} - \frac{1}{2}\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2) \|u(\cdot, t)\|_{L^2}^2 \\
    &\hspace{2em}+ \frac{1}{2}\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2 \big(2M\sqrt{b-a}\|u(\cdot, t) - M\|_{L^2}^2 + M^2(b-a)\big) \\
    &\hspace{2em}+ \frac{1}{2C^2} \|u(\cdot, t) - M\|_{L^2}^2 \\
    &\leq \chi^2 M^2(b-a)\|v_x(\cdot, t)\|_{L^\infty}^2 \left(\frac{C^2\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2}{1 - 2\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2} + \frac{1}{2}\right) \hspace{2em}+ \frac{1}{2C^2} \|u(\cdot, t) - M\|_{L^2}^2.
\end{align*}
\]

Here, we set \( K(t) := \frac{C^2\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2}{1 - 2\chi^2\|v_x(\cdot, t)\|_{L^\infty}^2} + \frac{1}{2} \). From the above arguments, we have the following inequality:
\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t) - M\|_{L^2}^2 \leq \chi^2 M^2(b-a)\|v_x(\cdot, t)\|_{L^\infty}^2 K(t) - \frac{1}{4C^2} \|u(\cdot, t) - M\|_{L^2}^2. \tag{44}
\]

From (44) and the assumption \( \lim_{t \to \infty} \|v_x(\cdot, t)\|_{L^\infty} = 0 \), by applying Proposition 4.6, we find
\[
\|u(\cdot, t) - M\|_{L^2}^2 \to 0 \quad (t \to \infty).
\]

Then, we shall prove (27). We denote by \((\tilde{u}, \tilde{v})\) the solution of (KS*), and let
\[
    w(x, t) := u(x, t) - \tilde{u}(x, t), \quad z(x, t) := v(x, t) - \tilde{v}(x, t).
\]

Then, we have easily the following system (*)
\[
\begin{cases}
    w_t = w_{xx} - \chi(uxv_x + uw_{xx}) & \text{in } I \times (0, \infty), \tag{45.1} \\
    z_t = z_{xx} - \gamma z + \alpha w & \text{in } I \times (0, \infty), \tag{45.2} \\
    w_x(a, t) = w_x(b, t) = z_x(a, t) = z_x(b, t) = 0 & \text{in } (0, \infty), \tag{45.3} \\
    w(x, 0) = w(x), \quad z(x, 0) = z(x) = 0 & \text{in } I.
\end{cases}
\]
Here, we are paying our attention to (45.2). As well as (42) and (44), we have
\[
\frac{1}{2} \frac{d}{dt} \int_I z^2 \, dx = - \int_I z_x^2 \, dx - \gamma \int_I z^2 \, dx + \alpha \int_I zw \, dx. \tag{46}
\]
We define
\[
F(t) := (\int_I z^2 \, dx)^{\frac{1}{2}} = \|z(\cdot, t)\|_{L^2}.
\]
Then the lefthand side of (46) is written as follows:
\[
\frac{1}{2} \frac{d}{dt} \int_I z^2 \, dx = \frac{1}{2} \frac{d}{dt} \{F(t)\}^2 = F(t)F'(t).
\]
Furthermore, we set
\[
G(t) := (\int_I w^2 \, dx)^{\frac{1}{2}} = \|w(\cdot, t)\|_{L^2}.
\]
By the Hölder inequality the right-hand side of (46) becomes
\[
- \int_I z_x^2 \, dx - \gamma \int_I z^2 \, dx + \alpha \int_I zw \, dx
\leq -\gamma\|z(\cdot, t)\|_{L^2}^2 + \alpha\|z(\cdot, t)\|_{L^2} \|w(\cdot, t)\|_{L^2}
\leq -\gamma\{F(t)\}^2 + \alpha F(t) G(t).
\]
That is,
\[
F(t)F'(t) \leq -\gamma\{F(t)\}^2 + \alpha F(t) G(t).
\]
Thus we obtain the following inequality:
\[
F'(t) \leq -\gamma F(t) + \alpha G(t). \tag{47}
\]
Note that (see. (26), (35))
\[
G(t) = \|w(\cdot, t)\|_{L^2} = \|(u-\tilde{u})(\cdot, t)\|_{L^2} \leq \|u(\cdot, t) - M\|_{L^2} + \|\tilde{u}(\cdot, t) - M\|_{L^2} \to 0 \ (t \to \infty),
\]
hence by Proposition 4.6, we have
\[
F(t) = \|z(\cdot, t)\|_{L^2} = \|(v - \tilde{v})(\cdot, t)\|_{L^2} \to 0 \ (t \to \infty).
\]
Moreover, since for bounded interval \(I\), \(L^\infty\) is continuously embedded in \(L^2\). By (36), it holds that
\[
\|\tilde{v}(\cdot, t) - \frac{\alpha M}{\gamma}\|_{L^2} \to 0 \ (t \to \infty).
\]
By combining the above two, we get
\[
\|v(\cdot, t) - \frac{\alpha M}{\gamma}\|_{L^2} \leq \|(v - \tilde{v})(\cdot, t)\|_{L^2} + \|\tilde{v}(\cdot, t) - \frac{\alpha M}{\gamma}\|_{L^2} \to 0 \ (t \to \infty).
\]
We have completed the proof of Theorem 4.7. \(\blacksquare\)
4.3 Examples of numerical calculation

As we showed in Theorem 4.7, the classical solutions $u$ and $v$ of (KS) with sufficiently small chemotaxis converge to the constants. Here, we introduce some examples of numerical calculations. From these examples, although it is heuristic, we give the following conjecture.

**Conjecture 4.10** Let $\overline{u}, \overline{v} \in W^{1,2}(I)$ and $\inf_{x \in I} \overline{u} > 0$, $\inf_{x \in I} \overline{v} > 0$, and $(u, v)$ be the classical solution of (KS). Assume that $\chi$ is small enough. Then, for any $x \in I$, the following hold:

$$\lim_{t \to \infty} u(x, t) = M,$$

$$\lim_{t \to \infty} v(x, t) = \frac{\alpha M}{\gamma}.$$ 

**Example 4.11** Let $\alpha = 2$, $\gamma = 3$, $a = 0$, $b = \pi$, $\overline{u}(x) = 3 - \cos 2x$, $\overline{v}(x) = 3$, $\chi = 1$. As we see in Figure 5, the solutions $u$ and $v$ converge to some constants, respectively.

**Example 4.12** Let $\alpha, \gamma, a, \overline{u}, \overline{v}(x)$ be same as Example 4.11, but let $\chi = \frac{5}{4}$. As we see in Figure 6, the solutions $u$ and $v$ do not converge to some constants.

![Figure 5 Result of the numerical computation of Example 4.11](image-url)

The figure of the left-hand side is the graph of $u = u(x, t)$, and the right-hand side is the graph of $v = v(x, t)$. 

26
Figure 6 Result of the numerical computation of Example 4.12
The figure of the left-hand side is the graph of $u = u(x,t)$, and the right-hand side is the graph of $v = v(x,t)$. 
Construction of a unique mild solution of one-dimensional Keller-Segel systems with uniformly elliptic operators having variable coefficients

5.1 Preliminaries

In this chapter, we denote by \( L^r \equiv L^r(\mathbb{R}) \), the Banach space with the norm \( \|u\|_{L^r} = \left( \int_{\mathbb{R}} |u(x)|^r \, dx \right)^{\frac{1}{r}} \) for \( 1 \leq r \leq \infty \). Firstly, let \( L \) and \( H \) be symmetric operators on \( S(\mathbb{R}) \) such that

\[ L \phi := \rho \phi_{xx} + \rho' \phi_x, \quad H \phi := \eta \phi_{xx} + \eta' \phi_x, \quad \text{for} \quad \phi \in S(\mathbb{R}), \]

where \( \rho \) and \( \eta \in C^\infty(\mathbb{R}) \) are given functions, and \( S(\mathbb{R}) \) is the Schwartz space of rapidly decreasing smooth functions. Furthermore, we assume \( L \) and \( H \) to be uniform elliptic operators, namely there exist some constants \( \lambda_1, \lambda_2 > 1 \) with

\[ \lambda_1 \geq \rho(x) \geq \frac{1}{\lambda_1} > 0, \quad \lambda_2 \geq \eta(x) \geq \frac{1}{\lambda_2} > 0, \quad \forall x \in \mathbb{R}. \]

The symmetric operators \( L \) and \( H \) correspond with the symmetric forms \( E_L \) and \( E_H \), respectively on \( S(\mathbb{R}) \) such that

\[ E_L(\varphi, \phi) \equiv \int_{\mathbb{R}} \varphi(x) \rho(x) \phi_x(x) \, dx = -\int_{\mathbb{R}} \varphi(x)(L\phi)(x) \, dx, \]

\[ E_H(\varphi, \phi) \equiv \int_{\mathbb{R}} \varphi(x) \eta(x) \phi_x(x) \, dx = -\int_{\mathbb{R}} \varphi(x)(H\phi)(x) \, dx. \]

Next, we extend \( L \) and \( H \) on \( S(\mathbb{R}) \) to self-adjoint operators on \( L^2(\mathbb{R}) \). By (49), \( E_L \) and \( E_H \) are closable on \( L^2(\mathbb{R}) \) (see e.g. [11] and Section II-2 of [19]), and we denote henceforth by \( E_L \) and \( E_H \) their closure respectively (thus using the same symbols). It is known that a closed semi-bounded quadratic form is the quadratic form of a unique self-adjoint operator (see Reed and Simon [27, Theorem VIII. 15]). We define the negative definite self-adjoint operators \( L \) and \( H \) (using the same symbols as (48) ) that correspond with the closed form \( E_L \) and \( E_H \) respectively, the domain of which are both the Sobolev space \( H^2(\mathbb{R}) \) (see. e.g. exercise 2.3.1 in Section 2.3 of [11]). Then the ultra-contractive semigroups \( e^{tL}, e^{tH} \) from \( L^1(\mathbb{R}) \) to \( L^\infty(\mathbb{R}) \) can be defined (see Stroock [28]).

In Section 5.4, we use the following estimates (see Fukushima [11], Gross [13], Ma and Röckner [19], Mizohata [22] and Stroock [28], Yosida [39]).

**Proposition 5.1** Let \( L \) be the non-positive definite self-adjoint operator defined above. Then, the followings hold (the corresponding results hold also for the self-adjoint operator \( H \)).

28
(i) (The contraction property, Lemma I.0.4 of [28])
For any \( \phi \in L^q(\mathbb{R}) \) \( (1 \leq q < \infty) \) and \( t > 0 \), it holds that \( e^{tL} \phi \in L^q(\mathbb{R}) \), and
\[
\|e^{tL} \phi\|_{L^q} \leq \|\phi\|_{L^q} \quad (1 \leq q < \infty).
\] (50)

(ii) (I.0.5 of [28])
For any \( \phi \in L^2(\mathbb{R}) \) and \( t > 0 \), it holds that \( \partial_x e^{tL} \phi \in L^2(\mathbb{R}) \), more precisely,
\[
\|\partial_x e^{tL} \phi\|_{L^2} \leq \frac{\lambda_1^2}{t^2} \|\phi\|_{L^2},
\] (51)
where \( \lambda_1 \) is given by (49).

(iii) (The ultra contractivity, the formula above (I.1.4) of [28])
There exists a constant \( l = l(\lambda_1) > 0 \), depending only on \( \lambda_1 \) given by (49), such that for any \( \phi \in L^1(\mathbb{R}) \) and \( t > 0 \), it holds that \( e^{tL} \phi \in L^\infty(\mathbb{R}) \), and
\[
\|e^{tL} \phi\|_{L^\infty} \leq \frac{l}{t^2} \|\phi\|_{L^1}.
\] (52)

(iv) (p.320 of [28])
There exists a constant \( k = k(\lambda_1) > 0 \), depending only on \( \lambda_1 \) given by (49), such that for any \( \phi \in L^1(\mathbb{R}) \) and \( t > 0 \), it holds that \( e^{tL} \phi \in L^2(\mathbb{R}) \), and
\[
\|e^{tL} \phi\|_{L^2} \leq \frac{k}{t^2} \|\phi\|_{L^1}.
\] (53)

(v) (I.0.10 of [28], see [11], [13] and Definition 4.1 and Proposition 4.3 of [19])
Let \( e^{tL}, t > 0 \) be Markovian semigroups. For any \( \phi \in L^\infty(\mathbb{R}) \) and \( t > 0 \), it holds that \( e^{tL} \phi \in L^\infty(\mathbb{R}) \), and
\[
\|e^{tL} \phi\|_{L^\infty} \leq \|\phi\|_{L^\infty}.
\] (54)

Since the domain of the negative definite self-adjoint operator \( H \) is the Sobolev space \( H^2(\mathbb{R}) \) (see the discussions given for the symmetric forms \( \mathcal{E}_L \) and \( \mathcal{E}_H \)), it holds that \( (\gamma - H)^{-1}(L^2(\mathbb{R})) = H^2(\mathbb{R}) \) and we have the inequalities corresponding to the resolvents, which play the essential roles in the proof of Theorem 5.3. Let \( \alpha > 0 \) and \( \gamma > 0 \) be some given constants, and for \( u \in L^2(\mathbb{R}) \) let \( v = \alpha(\gamma - H)^{-1}u \in H^2(\mathbb{R}) \) (for the general discussions, see e.g., Theorem 3.24 of [19]). The Sobolev embedding theorem implies
\[
\|v\|_{H^2} \leq C\|u\|_{L^2}, \quad \|v\|_{L^\infty} \leq C\|u\|_{L^2}, \quad \|\partial_x v\|_{L^\infty} \leq C\|u\|_{L^2},
\] (55)
where \( C = C(\alpha, \gamma, \lambda_2) > 0 \) is a constant depending only on \( \alpha, \gamma \) and \( \lambda_2 \).
5.2 Generalized Keller-Segel system

Here, we generalize the (KS') which has uniform elliptic operators $L$ and $H$ having variable coefficients $\rho$ and $\eta$, where $L$, $H$, $\rho$ and $\eta$ are given in Section 5.1. We call our system as generalized Keller-Segel system (GKS). We discuss the following:

\begin{align*}
\partial_t u &= Lu - \chi \partial_x(u \eta \partial_x v) & \text{in } \mathbb{R} \times (0, \infty), \\
0 &= Hv - \gamma v + \alpha u & \text{in } \mathbb{R} \times (0, \infty), \\
u(x,0) &= \overline{\pi}(x) \geq 0 & \text{in } \mathbb{R}
\end{align*}

(56)

for $\overline{\pi} \in \mathcal{S}(\mathbb{R})$, where $\chi > 0, \alpha > 0$ and $\gamma > 0$ are some given constants. Let $W$ be a Banach space and $J$ be an interval in $\mathbb{R}$. Then $L^p(J; W)$ is $W$-valued $L^p$ space in $J$. We present Banach spaces $X$ and $Y$ which are used to state our main results. For each given $0 < T < 1$, let $X$ and $Y$ be the Banach spaces defined by

\begin{align*}
X := \{ u \in L^\infty(0,T; L^2(\mathbb{R})) \cap L^\infty(0,T; L^\infty(\mathbb{R})) ; \ t^{\frac{1}{2}}(\partial_t u) \in L^\infty(0,T; L^2(\mathbb{R})) \}, \\
Y := L^\infty(0,T; H^2(\mathbb{R})).
\end{align*}

(57)

The norms of $X$ and $Y$ are given as follows:

$$
\|u\|_X := \sup_{0 < t < T} \|u(t)\|_{L^2} + \sup_{0 < t < T} \|u(t)\|_{L^\infty} + \sup_{0 < t < T} t^{\frac{1}{2}} \|\partial_x u(t)\|_{L^2},
$$

$$
\|v\|_Y := \sup_{0 < t < T} \|v(t)\|_{H^2} = \sup_{0 < t < T} \|v(t)\|_{L^2} + \|\partial_x v(t)\|_{L^2} + \|\partial_{xx} v(t)\|_{L^2}^{\frac{1}{2}}.
$$

Lastly in this section, we then set the definition of mild solution $(u, v)$ for (GKS) as follows:

**Definition 5.2** Let $X$ and $Y$ be the Banach spaces defined by (56) and (57) respectively. Two symmetric operators $L$, $H$ and function $\eta$ are given above. A pair of measurable functions $(u, v)$, $u \in X$ and $v \in Y$, is a mild solution of (GKS) on $\mathbb{R} \times (0,T)$ if it satisfies

\begin{align*}
\begin{cases}
    u &= e^{tL} \overline{\pi} - \chi \int_0^t e^{(t-\tau)L} \partial_x(u(\tau) \eta \partial_x v(\tau)) \, d\tau & \text{in } \mathbb{R} \times (0,T), \\
v &= \alpha(\gamma - H)^{-1} u & \text{in } \mathbb{R} \times (0,T), \\
u(x,0) &= \overline{\pi}(x)(\geq 0) & \text{in } \mathbb{R},
\end{cases}
\end{align*}

where $\chi > 0, \alpha > 0$ and $\gamma > 0$ are the constants appearing in (GKS).
5.3 Main results

Let two symmetric operators $L$, $H$ and a function $\eta$ be given in Section 5.2. Firstly, we define the sequences of successive approximations $\{u_n\}, \{v_n\}$ by making use of the Markovian semi-group $e^{tL}$ (cf. (54)) as follows:

$$
\begin{cases}
  u_1(t) = e^{tL}\overline{u}, \\
  u_{n+1}(t) = e^{tL}\overline{u} - \chi \int_0^t e^{(t-\tau)L}\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) \, d\tau, \\
  v_n(t) = \alpha (\gamma - H)^{-1} u_n(t), \quad n = 1, 2, \ldots
\end{cases}
$$

(58)

Our results are as follows:

Theorem 5.3 For a given function $\overline{u} \in S(\mathbb{R})$, let $\{u_n\}, \{v_n\}$ be the sequences defined by (58), and $X, Y$ be the Banach spaces given by (56) and (57), respectively. Then, there exists a positive constant $T$ which depends on $\chi, \alpha, \gamma, \overline{u}, \lambda_1, \lambda_2$ such that $u_* := \lim_{n \to \infty} u_n$ and $v_* := \lim_{n \to \infty} v_n$ with $(u_*, v_*) \in X \times Y$ is the mild solution of (GKS) on $\mathbb{R} \times (0, T)$. Moreover, the solution $(u_*, v_*)$ with the initial state $u_*(x, 0) = \overline{u}(x)(\geq 0)$ is unique.

Theorem 5.4 Corresponding statements as in Theorem 5.3 hold for any non-negative initial function $\overline{u} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

5.4 Proof of the existence part of Theorem 5.3

To prove the existence part of Theorem 5.3, we go through the three steps.

STEP 1.

For each $T \in (0, 1)$, by mathematical induction we shall show that there exist three sequences of numbers $\{A_n\}, \{B_n\}, \{C_n\}$ satisfying

$$
\sup_{0 < t < T} \|u_n(t)\|_{L^2} \leq A_n, \\
\sup_{0 < t < T} \|u_n(t)\|_{L^\infty} \leq B_n, \\
\sup_{0 < t < T} t^{\frac{1}{2}} \|\partial_x u_n(t)\|_{L^2} \leq C_n.
$$

(59) (60) (61)

Since $\overline{u} \in S(\mathbb{R})$, we notice that $\overline{u} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By the norm inequalities (50), (51) and (54) and the approximations (58), we have

$$
\|u_1(t)\|_{L^2} \leq \|\overline{u}\|_{L^2}, \quad \|u_1(t)\|_{L^\infty} \leq \|\overline{u}\|_{L^\infty}, \quad t^{\frac{1}{2}} \|\partial_x u_1(t)\|_{L^2} \leq \lambda_1^{\frac{1}{2}} \|\overline{u}\|_{L^2},
$$

where $\lambda_1$ is given by (49). We define

$$
A_1 := \|\overline{u}\|_{L^2}, \quad B_1 := \|\overline{u}\|_{L^\infty}, \quad C_1 := \lambda_1^{\frac{1}{2}} \|\overline{u}\|_{L^2}.
$$

(62)
Thus, (59), (60) and (61) hold for \( n = 1 \). Next, assume that (59), (60) and (61) are true for some \( n \). Then we will show that this implies that they are also true for \( n + 1 \), by the following procedures i), ii), and iii).

i) Evaluation of \( \|u_{n+1}(t)\|_{L^2} \).

By the norm inequality (53) and the approximations (58), we have

\[
\|u_{n+1}(t)\|_{L^2} \leq \|e^{tL}u_n\|_{L^2} + \chi \int_0^t \|e^{(t-\tau)L} \partial_x (u_n(\tau) \eta \partial_x v_n(\tau))\|_{L^2} \, d\tau \\
\leq A_1 + \chi \int_0^t \frac{k}{(t-\tau)^{\frac{3}{4}}} \|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau))\|_{L^1} \, d\tau,
\]

where \( k = k(\lambda_1) \) is the constant given by (53). Note first that

\[
\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) = \partial_x u_n(\tau) \eta \partial_x v_n(\tau) + u_n(\tau)H v_n(\tau).
\]

Applying the Hölder inequality, we obtain that

\[
\|\partial_x u_n(\tau) \eta \partial_x v_n(\tau)\|_{L^1} \leq \|\partial_x u_n(\tau)\|_{L^1} \cdot \|\partial_x v_n(\tau)\|_{L^\infty} + \|u_n(\tau)\|_{L^2} \|H v_n(\tau)\|_{L^2} =: I_1 + I_2.
\]

Here, consider the inequalities (49), (55). By the assumptions of the induction for \( A_n \) and \( C_n \) concerning (59) and (61), we have

\[
I_1 = \|\partial_x u_n(\tau)\|_{L^2} \|\eta\|_{L^\infty} \|\partial_x v_n(\tau)\|_{L^2} \leq \tau^{-\frac{1}{2}} \lambda_2 CA_n C_n.
\]

In addition, we know that \( H v_n(\tau) = \gamma v_n(\tau) - \alpha u_n(\tau) \) from the approximations (58), it holds that

\[
I_2 = \|u_n(\tau)\|_{L^2} \|H v_n(\tau)\|_{L^2} \leq \|u_n(\tau)\|_{L^2} (\|\gamma v_n(\tau)\|_{L^2} + \alpha \|u_n(\tau)\|_{L^2}) \leq (\gamma C + \alpha)A_n^2,
\]

where \( C = C(\alpha, \gamma, \lambda_2) \). Thus, by (65) and the estimates above, we see that

\[
\|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau))\|_{L^1} \leq I_1 + I_2 \leq C \lambda_2 A_n C_n \tau^{-\frac{1}{2}} + (\gamma C + \alpha)A_n^2.
\]

Substituting (66) into (63), we get

\[
\|u_{n+1}(t)\|_{L^2} \leq A_1 + \chi \int_0^t k CA_n C_n \left( \frac{3}{4} \right) \frac{1}{(t-\tau)^{\frac{3}{4}}} \, d\tau + \chi \int_0^t k (\gamma C + \alpha) A_n^2 \left( \frac{3}{4} \right) \frac{1}{(t-\tau)^{\frac{3}{4}}} \, d\tau \\
\leq A_1 + \chi k C \lambda_2 A_n C_n t^{\frac{1}{4}} B\left(\frac{3}{4}, \frac{1}{2}\right) + \chi k (\gamma C + \alpha) A_n^2 \frac{4}{3} t^{\frac{1}{4}},
\]

where \( B(p, q) (p, q > 0) \) denotes the Beta function. Define

\[
D_1 := \max\{\chi k C \lambda_2 B\left(\frac{3}{4}, \frac{1}{2}\right), \frac{4}{3} \chi k (\gamma C + \alpha)\}.
\]
Then, from (67) we consequently get the following evaluation.

\[
\sup_{0 < t < T} \|u_{n+1}(t)\|_{L^2} \leq A_1 + D_1 A_n C_n T^{\frac{1}{2}} + D_1 A_n^2 T^{\frac{3}{2}}. \tag{68}
\]

ii) Evaluation of \(\|u_{n+1}(t)\|_{L^\infty}\).

By the inequalities (52), (66) and the approximations (58), we have

\[
\|u_{n+1}(t)\|_{L^\infty} \leq \|e^{tL}u_{n}\|_{L^\infty} + \chi \int_0^t \|e^{(t-\tau)L} \partial_x (u_n(\tau) \eta \partial_x v_n(\tau))\|_{L^\infty} \, d\tau
\]

\[
\leq B_1 + \chi \int_0^t l (t - \tau)^{-\frac{1}{2}} \|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau))\|_{L^1} \, d\tau.
\]

\[
\leq B_1 + \chi \int_0^t \{ l C \lambda_2 A_n C_n^2 \tau^{-\frac{1}{2}} (t - \tau)^{-\frac{1}{2}} + l (\gamma C + \alpha) A_n^2 (t - \tau)^{-\frac{1}{2}} \} \, d\tau
\]

\[
= B_1 + \chi l C \lambda_2 A_n C_n B \left( \frac{1}{2}, \frac{1}{2} \right) + \chi l (\gamma C + \alpha) A_n^2 2t^\frac{1}{2}
\]

\[
\leq B_1 + \chi l C \lambda_2 A_n C_n \pi + 2\chi l (\gamma C + \alpha) A_n^2 T^{\frac{1}{2}},
\]

where \(l = l(\lambda_1)\) is the constant given by (52). Define

\[
D_2 := \max\{\chi l C \lambda_2 \pi, 2\chi l (\gamma C + \alpha)\}.
\]

Then we obtain the following estimate:

\[
\sup_{0 < t < T} \|u_{n+1}(t)\|_{L^\infty} \leq B_1 + D_2 A_n C_n + D_2 A_n^2 T^{\frac{1}{2}}. \tag{69}
\]

iii) Evaluation of \(t^{\frac{1}{2}} \|\partial_x u_{n+1}(t)\|_{L^2}\).

By the inequality (51) and the approximations (58), we have

\[
\|\partial_x u_{n+1}(t)\|_{L^2} \leq \chi \int_0^t \|\partial_x e^{(t-\tau)L} \partial_x (u_n(\tau) \eta \partial_x v_n(\tau))\|_{L^2} \, d\tau
\]

\[
\leq \chi \int_0^t \frac{\lambda_1^\frac{1}{2}}{(t - \tau)^{\frac{1}{2}}} \|\partial_x (u_n(\tau) \eta \partial v_n(\tau))\|_{L^2} \, d\tau. \tag{70}
\]

Then, by the equation (64), we get

\[
\|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau))\|_{L^2} \leq \|\partial_x u_n(\tau) \eta \partial v_n(\tau)\|_{L^2} + \|u_n(\tau) H v_n(\tau)\|_{L^2} =: J_1 + J_2. \tag{71}
\]

Consider the inequalities (49), (55). By the assumptions of the induction for \(A_n, B_n\) and \(C_n\) concerning (59) – (61), we have

\[
J_1 \leq \|\partial_x u_n(\tau)\|_{L^2} \|\eta\|_{L^\infty} \|\partial_x v_n(\tau)\|_{L^\infty} \leq C \lambda_2 A_n C_n \tau^{-\frac{1}{2}},
\]

\[
J_2 \leq \|u_n(\tau)\|_{L^\infty} \|H v_n(\tau)\|_{L^2} \leq B_n (\gamma \|v_n(\tau)\|_{L^2} + \alpha \|u_n(\tau)\|_{L^2}) \leq (\gamma C + \alpha) A_n B_n.
\]
Thus by (71) and the estimates above, we obtain
\[ \| \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^2} \leq C\lambda A_n C_n \tau^{-\frac{1}{2}} + (\gamma C + \alpha) A_n B_n. \] (72)

Therefore, substituting (72) into (70), we see that
\[ \| \partial_x u_{n+1}(t) \|_{L^2} \leq \chi \int_0^t \frac{\lambda^\frac{1}{2}}{(t - \tau)^{\frac{1}{2}}} (C\lambda A_n C_n \tau^{-\frac{1}{2}} + (\gamma C + \alpha) A_n B_n) \, d\tau \]
\[ \leq \chi C \lambda^\frac{1}{2} A_n C_n \lambda_2 B \left( \frac{1}{2}, \frac{1}{2} \right) + \chi \int_0^t \frac{\lambda^\frac{1}{2}}{(t - \tau)^{\frac{1}{2}}} (\gamma C + \alpha) A_n B_n \, d\tau \]
\[ \leq \chi C A_n C_n \lambda^\frac{1}{2} \lambda_2 \pi + 2\chi (\gamma C + \alpha) \lambda^\frac{1}{2} A_n B_n t^\frac{1}{2}. \] (73)

Define
\[ D_3 := \max\{\chi C \lambda^\frac{1}{2} \lambda_2 \pi, \ 2\chi (\gamma C + \alpha) \lambda^\frac{1}{2}\}. \]

Then from (73), we get the following:
\[ \sup_{0 < t < T} t^\frac{1}{2} \| u_{n+1}(t) \|_{L^2} \leq D_3 A_n C_n T^\frac{1}{2} + D_3 A_n B_n T. \] (74)

Further, define
\[ D_0 := \max\{A_1, B_1, C_1\}, \] (75)
\[ D_* := \max\{D_1, D_2, D_3\}. \] (76)

Furthermore, set
\[ A_{n+1} := D_0 + D_* A_n C_n T^\frac{1}{2} + D_* A_n^2 T^\frac{3}{2}, \] (77)
\[ B_{n+1} := D_0 T^{-\frac{1}{2}} + D_* A_n C_n + D_* A_n^2 T^\frac{1}{2}, \] (78)
\[ C_{n+1} := D_0 + D_* A_n C_n T^\frac{1}{2} + D_* A_n B_n T. \] (79)

Remark that $D_0 = D_0(\bar{\pi}, \lambda_1)$, and $D_* = D_*(\chi, \alpha, \gamma, \lambda_1, \lambda_2)$ are constants depending only on $\bar{\pi}, \lambda_1$, and $\chi, \alpha, \gamma, \lambda_1, \lambda_2$, respectively. Consider (68), (69) and (74). Use (77) – (79). Then, we have proved that the estimates (59), (60), and (61) hold for $n + 1$. Note that for $0 < T < 1$, it holds that $T^\frac{1}{2} < T^\frac{3}{2}$ in (74) and (79).

STEP 2.

In the following, by the inductive method, we shall prove that the sequences $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ defined by (62), (77), (78), and (79) are bounded above and non-decreasing.

(1) Proof of the non-decreasing property.

It is obvious that
\[ A_2 = D_0 + D_* A_1 C_1 T^\frac{1}{2} + D_* A_1^2 T^\frac{3}{2} \geq D_0 \geq A_1, \]
\[ B_2 = D_0 T^{-\frac{1}{4}} + D_1 A_1 + D_2 A_1 T^{\frac{1}{4}} \geq D_0 T^{-\frac{1}{4}} \geq B_1, \]
\[ C_2 = D_0 + D_1 A_1 T^{\frac{3}{4}} + D_2 A_1 B_1 T \geq D_0 \geq C_1. \]
Suppose that \( A_n \leq A_{n+1}, B_n \leq B_{n+1}, C_n \leq C_{n+1} \) for some \( n \). Then we have
\[ A_{n+2} = D_0 + D_1 A_{n+1} C_{n+1} T^{\frac{3}{4}} + D_2 A_{n+1} T^{\frac{3}{4}} \geq D_0 + D_1 A_n C_n T^{\frac{3}{4}} + D_2 A_n^2 T^{\frac{9}{4}} = A_{n+1}. \]
\[ B_{n+2} = D_0 T^{-\frac{1}{4}} + D_1 A_{n+1} C_{n+1} + D_2 A_{n+1} T^{\frac{1}{4}} \geq D_0 T^{-\frac{1}{4}} + D_1 A_n C_n + D_2 A_n T^{\frac{1}{4}} = B_{n+1}. \]
\[ C_{n+2} = D_0 + D_1 A_{n+1} C_{n+1} T^{\frac{3}{4}} + D_2 A_{n+1} B_{n+1} T \geq D_0 + D_1 A_n C_n T^{\frac{3}{4}} + D_2 A_n B_n T = C_{n+1}. \]
Hence, we have proved that the sequences \( \{A_n\}, \{B_n\} \) and \( \{C_n\} \) are non-decreasing.

(II) Proof of the upper bound.

We introduce three equations with respect to \( x, y, \) and \( z \).
\[ x = D_0 + D_1 x z T^{\frac{3}{4}} + D_2 x^2 T^{\frac{9}{4}}, \quad (80) \]
\[ y = D_0 T^{-\frac{1}{4}} + D_1 x z + D_2 x^2 T^{\frac{3}{4}}, \quad (81) \]
\[ z = D_0 + D_1 x z T^{\frac{3}{4}} + D_2 x y T, \quad (82) \]
where \( D_0, D_1 \) are given in (75), (76), and \( 0 < T < 1 \). From (80) and (81), we have
\[ x = y T^{\frac{3}{4}}. \quad (83) \]
By substituting (83) into (82), we get
\[ z = D_0 + D_1 x z T^{\frac{3}{4}} + D_2 x^2 T^{\frac{9}{4}}. \quad (84) \]
Comparing (80) with (84), we notice that
\[ x = z. \quad (85) \]
Substituting (85) into (80), we obtain following quadratic equation for \( x \):
\[ x = D_0 + D_1 x^2 T^{\frac{3}{4}} + D_2 x^2 T^{\frac{9}{4}}, \]
that is,
\[ D_1 (T^{\frac{3}{4}} + T^{\frac{9}{4}}) x^2 - x + D_0 = 0. \quad (86) \]
The existence condition for the real solution for (86) is given by
\[ 1 - 4 D_1 (T^{\frac{3}{4}} + T^{\frac{9}{4}}) D_0 \geq 0. \quad (87) \]
This inequality holds for small \( T > 0 \). Then we let \( x_* \) be the smallest solution of (86) under the condition (87), that is,
\[ x_* := \frac{1 - \sqrt{1 - 4 D_1 (T^{\frac{3}{4}} + T^{\frac{9}{4}}) D_0}}{2 D_1 (T^{\frac{3}{4}} + T^{\frac{9}{4}})}. \quad (88) \]
Further define
\[ y_* := x_* T^{-1/4}, \quad z_* := x_* \]
Remark that \( x_*, y_* \) and \( z_* \) depend on \( T \). Here, the numbers \( x_*, y_* \) and \( z_* \) are the solutions of (80), (81) and (82). Then, by induction we can prove
\[ A_n \leq x_*, \quad B_n \leq y_*, \quad C_n \leq z_* \quad (90) \]
as follows: For \( n = 1 \), (90) is true, since
\begin{align*}
A_1 & \leq D_0 + D_1 x_* z_* T^{1/4} + D_2 x_*^2 T^{3/4} = x_* \quad (91) \\
B_1 & \leq D_0 T^{-1/4} + D_1 x_* z_* + D_2 x_*^2 T^{1/2} = y_* \quad (92) \\
C_1 & \leq D_0 + D_1 x_* z_* T^{1/4} + D_2 x_* y_* T = z_* \quad (93)
\end{align*}
Suppose that (90) holds for some \( n \). Then, it holds that
\begin{align*}
A_{n+1} &= D_0 + D_1 A_n C_n T^{1/4} + D_2 A_n^2 T^{3/4} \leq D_0 + D_1 x_* z_* T^{1/4} + D_2 x_*^2 T^{3/4} = x_* \quad (94) \\
B_{n+1} &= D_0 T^{-1/4} + D_1 A_n C_n + D_2 A_n^2 T^{1/2} \leq D_0 T^{-1/4} + D_1 x_* z_* + D_2 x_*^2 T^{1/2} = y_* \quad (95) \\
C_{n+1} &= D_0 + D_1 A_n C_n T^{1/4} + D_2 A_n B_n T \leq D_0 + D_1 x_* z_* T^{1/4} + D_2 x_* y_* T = z_*
\end{align*}
Hence, we have proved that the sequences \( \{A_n\}, \{B_n\} \) and \( \{C_n\} \) are bounded above. Therefore, there exist three numbers \( A_*, B_* \) and \( C_* \) such that
\begin{align*}
\lim_{n \to \infty} A_n &= A_* \leq x_* \\
\lim_{n \to \infty} B_n &= B_* \leq y_* \\
\lim_{n \to \infty} C_n &= C_* \leq z_*
\end{align*}
STEP 3.

We shall prove the existence of \( u_* \in X \) satisfying
\begin{align*}
\sup_{0 < t < T} \|u_n(t) - u_*(t)\|_{L^2} &\to 0 \quad (n \to \infty), \quad (91) \\
\sup_{0 < t < T} \|u_n(t) - u_*(t)\|_{L^\infty} &\to 0 \quad (n \to \infty), \quad (92) \\
\text{and} \quad \sup_{0 < t < T} t^{1/2} \|\partial_x u_n(t) - \partial_x u_*(t)\|_{L^2} &\to 0 \quad (n \to \infty). \quad (93)
\end{align*}
The proofs of (91), (92) and (93) can be carried out as follows: Define the sequence \( \{U_n\} \) by
\[ U_{n+1}(t) := u_{n+1}(t) - u_n(t) \quad (n \geq 1), \quad U_1(t) := u_1(t). \quad (94) \]
By the analogous discussion as that in STEP 1, we shall show that there exist sequences of numbers \( \{A_n\}, \{B_n\}, \) and \( \{C_n\} \) satisfying
\[
\sup_{0 < t < T} \|U_n(t)\|_{L^2} \leq \tilde{A}_n, \tag{95}
\]
\[
\sup_{0 < t < T} \|U_n(t)\|_{L^\infty} \leq \tilde{B}_n, \tag{96}
\]
and
\[
\sup_{0 < t < T} t^{\frac{3}{2}} \|\partial_t U_n(t)\|_{L^2} \leq \tilde{C}_n. \tag{97}
\]
Indeed, by induction these boundedness properties are investigated in the following.

Define
\[
\tilde{A}_1 := D_0, \quad \tilde{B}_1 := D_0 T^{-\frac{1}{4}}, \quad \tilde{C}_1 := D_0. \tag{98}
\]
We have that (95), (96) and (97) are true for \( n = 1 \). Suppose that (95), (96) and (97) are true for some \( n \). Then the definitions (58) and (94) imply
\[
U_{n+1}(t) = -\chi \int_0^t e^{(t-\tau)L} \left( \partial_x \left( u_n(\tau) \eta \partial_x v_n(\tau) \right) - \partial_x \left( u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau) \right) \right) d\tau. \tag{99}
\]
for \( n \geq 1 \). Note that we define \( u_0 \equiv 0 \) and \( v_0 \equiv 0 \).

i) Evaluation of \( \|U_{n+1}(t)\|_{L^2} \).

By the inequality (53) and the equation (99), we have
\[
\|U_{n+1}(t)\|_{L^2} \\
\leq \chi \int_0^t \| e^{(t-\tau)L} \left( \partial_x \left( u_n(\tau) \eta \partial_x v_n(\tau) \right) - \partial_x \left( u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau) \right) \right) \|_{L^2} d\tau \\
\leq \chi \int_0^t \frac{k}{(t-\tau)^{\frac{1}{4}}} \| \partial_x \left( u_n(\tau) \eta \partial_x v_n(\tau) \right) - \partial_x \left( u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau) \right) \|_{L^1} d\tau. \tag{100}
\]
Notice (62) – (65). Then we can estimate the integrand on the right-hand side of (100) as follows:
\[
\| \partial_x \left( u_n(\tau) \eta \partial_x v_n(\tau) \right) - \partial_x \left( u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau) \right) \|_{L^1} \\
= \| \partial_x u_n(\tau) \eta \partial_x v_n(\tau) + u_n(\tau) H v_n(\tau) - \partial_x u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau) - u_{n-1} H v_{n-1}(\tau) \|_{L^1} \\
\leq \| \partial_x u_n(\tau) \eta \partial_x v_n(\tau) \|_{L^1} + \| \partial_x u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau) \|_{L^1} \\
+ \| u_n(\tau)(H v_n - v_{n-1}(\tau)) \|_{L^1} + \| u_{n-1}(\tau)(H v_{n-1}(\tau)) \|_{L^1} \\
=: K_1 + K_2 + K_3 + K_4. \tag{101}
\]
Now, set \( V_n = v_n - v_{n-1} \). Then we see that
\[
V_n = \alpha(\gamma - H)^{-1} u_n - \alpha(\gamma - H)^{-1} u_{n-1} = \alpha(\gamma - H)^{-1} U_n.
\]
Consider the inequalities (55), (49) and (90). By the assumptions of the induction for $A_n, \tilde{C}_n$ concerning (95) and (97), we obtain the following estimates.

$K_1 = \|\partial_x u_n(\tau) \eta \partial_x (v_n - v_{n-1})(\tau)\|_{L^1}$
$\leq \|\partial_x u_n(\tau)\|_{L^2}\|\eta\|_{L^\infty}\|\partial_x v_n(\tau)\|_{L^2}$
$\leq C\lambda_2 C_n \tilde{A}_n \tau^{-\frac{1}{2}} \leq C\lambda_2 z^*_x \tilde{A}_n \tau^{-\frac{1}{2}}.$

$K_2 = \|\partial_x (u_n - u_{n-1})(\tau) \eta \partial_x v_{n-1}(\tau)\|_{L^1}$
$\leq \|\partial_x U_n(\tau)\|_{L^2}\|\eta\|_{L^\infty}\|\partial_x v_{n-1}(\tau)\|_{L^2}$
$\leq C\lambda_2 \tilde{C}_n A_{n-1} \tau^{-\frac{1}{2}} \leq C\lambda_2 x^*_x \tilde{C}_n \tau^{-\frac{1}{2}}.$

$K_3 = \|u_n(\tau) H(v_n - v_{n-1})(\tau)\|_{L^1}$
$\leq \|u_n(\tau)\|_{L^2}\|HV_n(\tau)\|_{L^2}$
$\leq A_n \|\gamma V_n(\tau) - \alpha U_n(\tau)\|_{L^2}$
$\leq (\gamma C + \alpha)x^*_x \tilde{A}_n.$

$K_4 = \|(u_n - u_{n-1})(\tau) H v_{n-1}(\tau)\|_{L^1}$
$\leq \|U_n\|_{L^2}\|Hv_{n-1}\|_{L^2}$
$\leq \tilde{A}_n \|\gamma v_{n-1} - \alpha u_{n-1}\|_{L^2}$
$\leq (\gamma C + \alpha)x^*_x \tilde{A}_n.$

By (101) and the estimates above, we have the following inequality.

$\|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau))\|_{L^1}$
$\leq C\lambda_2 (z^*_x \tilde{A}_n + x^*_x \tilde{C}_n) \tau^{-\frac{1}{2}} + 2(\gamma C + \alpha)x^*_x \tilde{A}_n.$ \hspace{1cm} (102)

By substituting (102) into (100), we get

$\|U_{n+1}(t)\|_{L^2} \leq \chi \int_0^t \frac{k}{(t - \tau)^{\frac{1}{2}}} \left( C\lambda_2 (z^*_x \tilde{A}_n + x^*_x \tilde{C}_n) \tau^{-\frac{1}{2}} + 2(\gamma C + \alpha)x^*_x \tilde{A}_n \right) d\tau$
$\leq \chi k C\lambda_2 (z^*_x \tilde{A}_n + x^*_x \tilde{C}_n) B \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) T^{\frac{1}{4}} + \frac{8}{3} \chi k (\gamma C + \alpha)x^*_x \tilde{A}_n T^{\frac{3}{4}}.$

This shows that

$\sup_{0 < t < T} \|U_{n+1}(t)\|_{L^2} \leq \tilde{D}_1 (z^*_x \tilde{A}_n + x^*_x \tilde{C}_n) T^{\frac{1}{4}} + \tilde{D}_1 x^*_x \tilde{A}_n T^{\frac{3}{4}}, \hspace{1cm} (103)$

where

$\tilde{D}_1 := \max\{\chi k C\lambda_2 B \left( \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \frac{8}{3} (\gamma C + \alpha)\}.$

ii) Evaluation of $\|U_{n+1}(t)\|_{L^\infty}.$
By the inequalities (52), (102) and the equation (99), we have

\[
\|U_{n+1}(t)\|_{L^\infty} \leq \chi \int_0^t \|e^{(t-\tau)L}(\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau)))\|_{L^\infty} d\tau \\
\leq \chi \int_0^t \frac{l}{(t-\tau)^{\frac{1}{2}}} \|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau))\|_{L^1} d\tau \\
\leq \chi \int_0^t \frac{l}{(t-\tau)^{\frac{1}{2}}} \left(C\lambda_2(z_\ast \tilde{A}_n + x_\ast \tilde{C}_n)\right) d\tau \\
\leq C\lambda_2 \pi (z_\ast \tilde{A}_n + x_\ast \tilde{C}_n) + 4(\gamma C + \alpha)x_\ast \tilde{A}_n t^{\frac{1}{2}}.
\]

Hence, we obtain the following estimate.

\[
\sup_{0 \leq t \leq T} \|U_{n+1}(t)\|_{L^\infty} \leq \tilde{D}_2 (z_\ast \tilde{A}_n + x_\ast \tilde{C}_n) + \tilde{D}_2 x_\ast \tilde{A}_n T^{\frac{1}{2}},
\]

where

\[
\tilde{D}_2 := \max\{C\lambda_2 \pi, 4(\gamma C + \alpha)\}.
\]

iii) Evaluation of \( t^{\frac{1}{2}} \|\partial_x U_{n+1}(t)\|_{L^2} \).

By the same discussion as (70), it holds that

\[
\|\partial_x U_{n+1}(t)\|_{L^2} \\
\leq \chi \int_0^t \|\partial_x e^{(t-\tau)L}(\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau)))\|_{L^2} d\tau \\
\leq \chi \int_0^t \frac{\lambda_2^{\frac{1}{2}}}{(t-\tau)^{\frac{1}{2}}} \|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau))\|_{L^2} d\tau. \tag{105}
\]

Then, by the analogous argument as (101), we then get

\[
\|\partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau))\|_{L^2} \\
= \|\partial_x u_n(\tau) \eta \partial_x v_n(\tau) + u_n(\tau) H v_n - \partial_x u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau) - u_{n-1} H v_{n-1}(\tau)\|_{L^2} \\
\leq \|\partial_x u_n(\tau) \eta \partial_x (v_n - v_{n-1})(\tau)\|_{L^2} + \|\partial_x (u_n - u_{n-1})(\tau) \eta \partial_x v_{n-1}(\tau)\|_{L^2} \\
+ \|u_n(\tau) H (v_n - v_{n-1})(\tau)\|_{L^2} + \|(u_n - u_{n-1})(\tau) H v_{n-1}(\tau)\|_{L^2} \\
=: L_1 + L_2 + L_3 + L_4. \tag{106}
\]

Furthermore, observe the inequalities (55), (49) and (90). Then consider the assumptions of the induction for \( \tilde{A}_n, \tilde{C}_n \) concerning (95) and (97), again. By the same discussion as for \( K_1 - K_4 \), we obtain the following estimates:

\[
L_1 = \|\partial_x u_n(\tau) \eta \partial_x (v_n - v_{n-1})(\tau)\|_{L^2} \\
\leq \|\partial_x u_n(\tau)\|_{L^2} \|\eta\|_{L^\infty} \|\partial_x V_n(\tau)\|_{L^\infty} \\
\leq C\|\partial_x u_n(\tau)\|_{L^2} \lambda_2 \|U_n(\tau)\|_{L^2} \\
\leq C\tau^{-\frac{1}{2}} C_n \lambda_2 \tilde{A}_n \leq C\lambda_2 z_\ast \tau^{-\frac{1}{2}} \tilde{A}_n.
\]
\[ L_2 = \| \partial_x (u_n - u_{n-1})(\tau) \eta \partial_x v_{n-1}(\tau) \|_{L^2} \]
\[ \leq \| \partial_x U_n(\tau) \|_{L^2} \| \eta \|_{L^\infty} \| \partial_x v_{n-1}(\tau) \|_{L^\infty} \]
\[ \leq C \| \partial_x U_n(\tau) \|_{L^2} \| \eta \|_{L^\infty} \| u_{n-1}(\tau) \|_{L^2} \]
\[ \leq C \tau^{-\frac{1}{2}} \tilde{C}_n \lambda_2 A_{n-1} \leq C \lambda_2 x_s \tau^{-\frac{1}{2}} \tilde{C}_n. \]

\[ L_3 = \| u_n(\tau) H(v_n - v_{n-1})(\tau) \|_{L^2} \]
\[ \leq \| u_n(\tau) \|_{L^\infty} \| Hv_n(\tau) \|_{L^2} \]
\[ \leq B_n(\gamma \| V_n(\tau) \|_{L^2} + \alpha \| U_n(\tau) \|_{L^2}) \]
\[ \leq y_s(\gamma C \| U_n(\tau) \|_{L^2} + \alpha \| U_n(\tau) \|_{L^2}) \leq (\gamma C + \alpha) y_s \tilde{A}_n. \]

\[ L_4 = \| (u_n - u_{n-1})(\tau) Hv_{n-1}(\tau) \|_{L^2} \]
\[ \leq \| U_n(\tau) \|_{L^\infty} \| Hv_{n-1}(\tau) \|_{L^2} \]
\[ \leq \tilde{B}_n(\gamma \| v_{n-1}(\tau) \|_{L^2} + \alpha \| u_{n-1}(\tau) \|_{L^2}) \]
\[ \leq \tilde{B}_n(\gamma C x_s + \alpha x_s) \leq (\gamma C + \alpha)(y_s \tilde{A}_n + x_s \tilde{B}_n). \]  

By (106) and the estimates above, it holds that
\[ \| \partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau)) \|_{L^2} \]
\[ \leq C \lambda_2 \tau^{-\frac{1}{2}} (z_s \tilde{A}_n + x_s \tilde{C}_n) + (\gamma C + \alpha)(y_s \tilde{A}_n + x_s \tilde{B}_n). \]  

(107)

Then, by substituting (107) into (105), we have
\[ \| \partial_x U_{n+1}(t) \|_{L^2} \]
\[ \leq \chi \int_0^t \frac{\lambda_2^{\frac{1}{2}}}{(t - \tau)^{\frac{1}{2}}} \| \partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x (u_{n-1}(\tau) \eta \partial_x v_{n-1}(\tau)) \|_{L^2} d\tau \]
\[ \leq \chi \int_0^t \frac{\lambda_2^{\frac{1}{2}}}{(t - \tau)^{\frac{1}{2}}} \left( C \lambda_2 \tau^{-\frac{1}{2}} (z_s \tilde{A}_n + x_s \tilde{C}_n) + (\gamma C + \alpha)(y_s \tilde{A}_n + x_s \tilde{B}_n) \right) d\tau \]
\[ \leq C \chi \lambda_2^{\frac{1}{2}} \lambda_2 B \left( \frac{1}{2}, \frac{1}{2} \right) (z_s \tilde{A}_n + x_s \tilde{C}_n) + \chi \lambda_2^{\frac{1}{2}} (\gamma C + \alpha)(y_s \tilde{A}_n + x_s \tilde{B}_n) t^{\frac{1}{2}}. \]

This shows that
\[ \sup_{0 < t < T} t^{\frac{1}{2}} \| \partial_x U_{n+1}(t) \|_{L^2} \leq \tilde{D}_3(z_s \tilde{A}_n + x_s \tilde{C}_n) T^{\frac{1}{2}} + \tilde{D}_3(y_s \tilde{A}_n + x_s \tilde{B}_n) T, \]  

(108)

where
\[ \tilde{D}_3 := \max\{C \chi \lambda_2^{\frac{1}{2}} \lambda_2 \pi, \chi \lambda_2^{\frac{1}{2}} (\gamma C + \alpha)\}. \]

We set \( \tilde{D}_s = \tilde{D}_s(\chi, \alpha, \gamma, \lambda_1, \lambda_2) \) as
\[ \tilde{D}_s := \max\{\tilde{D}_1, \tilde{D}_2, 2 \tilde{D}_3\}. \]
Further define
\[ \tilde{A}_{n+1} := \tilde{D}_s(z_\star \tilde{A}_n + x_\star \tilde{C}_n)T^{\frac{1}{2}} + \tilde{D}_s x_\star \tilde{A}_n T^{\frac{3}{2}}, \]  
(109)
\[ \tilde{B}_{n+1} := \tilde{D}_s(z_\star \tilde{A}_n + x_\star \tilde{C}_n) + \tilde{D}_s x_\star \tilde{A}_n T^2, \]  
(110)
\[ \tilde{C}_{n+1} := \tilde{D}_s(z_\star \tilde{A}_n + x_\star \tilde{C}_n)T^{\frac{1}{2}} + \frac{1}{2} \tilde{D}_s(y_\star \tilde{A}_n + x_\star \tilde{B}_n)T. \]  
(111)

Consider (103), (104) and (108). Use (109) – (111). Then, we have proved that the estimates (95), (96), and (97) hold for \( n + 1 \). Note that for \( 0 < T < 1 \), it holds that \( T^\frac{3}{2} < T^2 \) in (108) and (111).

The definitions (109) and (110) imply

\[ \tilde{B}_n = \tilde{A}_n T^{-\frac{1}{2}}. \]  
(112)

By substituting (112) and \( y_\star = x_\star T^{-\frac{1}{2}} \) (see (89)) into (111), we get

\[ \tilde{C}_{n+1} := \tilde{D}_s(z_\star \tilde{A}_n + x_\star \tilde{C}_n)T^{\frac{1}{2}} + \frac{1}{2} \tilde{D}_s(x_\star T^{-\frac{1}{2}}\tilde{A}_n + x_\star \tilde{A}_n T^{-\frac{1}{2}})T, \]

that is,

\[ \tilde{C}_{n+1} := \tilde{D}_s(z_\star \tilde{A}_n + x_\star \tilde{C}_n)T^{\frac{1}{2}} + \tilde{D}_s x_\star \tilde{A}_n T^{\frac{3}{2}}. \]  
(113)

From (109) and (113), we obtain

\[ \tilde{A}_n = \tilde{C}_n. \]  
(114)

Remark that (112) and (114) hold for \( n = 1 \) (see (98)). By (109) and (113), we see

\[ \tilde{A}_{n+1} = (\tilde{D}_s(x_\star + z_\star)T^{\frac{1}{2}} + \tilde{D}_s x_\star T^{\frac{3}{2}})\tilde{A}_n. \]  
(115)

The equation (115) indicates that \( \{\tilde{A}_n\} \) is a geometric sequence of common ratio \( \tilde{D}_s(x_\star + z_\star)T^{\frac{1}{2}} + \tilde{D}_s x_\star T^{\frac{3}{2}} \). Hence, under the condition that

\[ \tilde{D}_s(x_\star + z_\star)T^{\frac{1}{2}} + \tilde{D}_s x_\star T^{\frac{3}{2}} < 1, \]  
(116)

\( \tilde{A}_n \) converges to 0. Remember that \( x_\star = z_\star \) depends on \( T \); see (88) and (89). We have to certify that (116) holds for small \( T \). Indeed, by substituting (88) and (89) into (116), we have

\[ \tilde{D}_s \frac{1 - \sqrt{1 - 4\tilde{D}_s(T^{\frac{1}{2}} + T^2)}}{\tilde{D}_s(1 + T^2)} + \tilde{D}_s \frac{1 - \sqrt{1 - 4\tilde{D}_s(T^{\frac{1}{2}} + T^2)}}{2\tilde{D}_s(1 + T^2)} T^2 < 1. \]  
(117)

Since (117) holds for small \( T = T(D_0, D_s, \tilde{D}_s) = T(\chi, \alpha, \gamma, \pi, \lambda_1, \lambda_2) > 0 \), (116) also holds. Therefore, for small \( T \), it holds that

\[ \sup_{0 < t < T} \| (u_{n+1} - u_n)(t) \|_{L^2} = \sup_{0 < t < T} \| U_{n+1}(t) \|_{L^2} \leq \tilde{A}_{n+1} \to 0 \quad (n \to \infty). \]  
(118)
As well as (118), by means of (112) and (114), we obtain the following:

\[ \sup_{0 < t < T} \|(u_{n+1} - u_n)(t)\|_{L^\infty} \leq \tilde{B}_{n+1} \to 0 \ (n \to \infty), \]

\[ \sup_{0 < t < T} t^{\frac{1}{2}}\|\partial_x (u_{n+1} - u_n)(t)\|_{L^2} \leq \tilde{C}_{n+1} \to 0 \ (n \to \infty). \]

Thus the results above mean that \( \{u_n\} \) is a Cauchy sequence in the Banach space \( X \) given by (56). Indeed, set \( S_n := \sum_{k=1}^{n} \tilde{A}_k \). Since the number sequence \( \{S_n\} \) is convergent, it follows that \( \{S_n\} \) is also a Cauchy sequence. Then we have

\[ \sup_{0 < t < T} \|(u_{n+m} - u_n)(t)\|_{L^2} \leq \tilde{A}_{n+1} + \tilde{A}_{n+2} + \cdots \tilde{A}_{n+m} = S_{n+m} - S_n \to 0 \ (n, m \to \infty). \]

In the same way, it holds that

\[ \sup_{0 < t < T} \|(u_{n+m} - u_n)(t)\|_{L^\infty} \to 0 \ (n, m \to \infty), \]

and

\[ \sup_{0 < t < T} t^{\frac{1}{2}}\|\partial_x (u_{n+m} - u_n)(t)\|_{L^2} \to 0 \ (n, m \to \infty). \]

Therefore, there exists a \( u_* \in X \) such that

\[ \sup_{0 < t < T} \|(u_n - u_*)(t)\|_{L^2} \to 0 \ (n \to \infty), \]  

\[ \sup_{0 < t < T} \|(u_n - u_*)(t)\|_{L^\infty} \to 0 \ (n \to \infty), \]

\[ \sup_{0 < t < T} t^{\frac{1}{2}}\|\partial_x (u_n - u_*)(t)\|_{L^2} \to 0 \ (n \to \infty). \]

Consequently, the convergences (91), (92) and (93) are verified.

Finally, define

\[ v_* := \frac{1}{\alpha(\gamma - H)} v_*. \]  

Then we can certify that \((u_*, v_*)\) is a mild solution of (GKS) on \((0, T)\) by the following procedures i), ii) and iii). At first, by (11) we note that

\[ u_*(t) - \{e^{tL}u - \chi \int_{0}^{t} e^{(t-\tau)L} \partial_x u_*(\tau) \eta \partial_x v_*(\tau) d\tau \} \]

\[ = (u_* - u_{n+1})(t) + \chi \int_{0}^{t} e^{(t-\tau)L} \left( \partial_x u_*(\tau) \eta \partial_x v_*(\tau) - \partial_x (u_n(\tau) \eta \partial_x v_n(\tau)) \right) d\tau. \]  

i) We proceed to the evaluation of \( \|u_*(t) - \{e^{tL}u - \chi \int_{0}^{t} e^{(t-\tau)L} \partial_x u_*(\tau) \eta \partial_x v_*(\tau) d\tau \|_{L^2} \):
By (53) and (123), we have
\[
\|u_n(t) - \left\{ e^{t\overline{a}} - \chi \int_0^t e^{(t-\tau)L} \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \, d\tau \right\} \|_{L^2} \\
\leq \| (u_n - u_{n+1})(t) \|_{L^2} \\
+ \chi \int_0^t \| e^{(t-\tau)L} \left( \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \right) \|_{L^2} \, d\tau \\
\leq \| (u_n - u_{n+1})(t) \|_{L^2} \\
+ \chi \int_0^t \frac{k}{(t-\tau)^{\frac{3}{2}}} \| \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^1} \, d\tau. \tag{124}
\]

The integrand on the right-hand side of (124) is evaluated as follows:
\[
\| \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^1} \\
= \| \partial_x u_n(\tau) \eta \partial_x v_n(\tau) + u_n(\tau)H v_n(\tau) - \partial_x u_n(\tau) \eta \partial_x v_n(\tau) - u_nH v_n(\tau) \|_{L^1} \\
\leq \| \partial_x u_n(\tau) \eta \partial_x(v_n - v_n(\tau)) \|_{L^1} + \| \partial_x u_n(\tau) \eta \partial_x v_n(\tau) \|_{L^1} \\
+ \| u_n(\tau)(H v_n - v_n(\tau)) \|_{L^1} + \| (u_n - u_n)(\tau)H v_n(\tau) \|_{L^1} \\
=: M_1 + M_2 + M_3 + M_4. \tag{125}
\]

Here, by the inequality (55) and the convergences (119) – (121), each \( M_i \) is evaluated as follows: for any \( \tau \in (0, T) \),
\[
M_1 = \| \partial_x u_n(\tau) \eta \partial_x(v_n - v_n(\tau)) \|_{L^1} \\
\leq \| \partial_x u_n(\tau) \|_{L^2} \| \eta \|_{L^\infty} \| \partial_x(v_n - v_n(\tau)) \|_{L^2} \\
\leq C \| \partial_x u_n(\tau) \|_{L^2} \| \eta \|_{L^\infty} \| (u_n - u_n)(\tau) \|_{L^2} \to 0 \quad (n \to \infty).
\]
\[
M_2 = \| \partial_x(u_n - u_n)(\tau) \eta \partial_x v_n(\tau) \|_{L^1} \\
\leq \| \partial_x(u_n - u_n)(\tau) \|_{L^2} \| \eta \|_{L^\infty} \| \partial_x v_n(\tau) \|_{L^2} \to 0 \quad (n \to \infty).
\]
\[
M_3 = \| u_n(\tau)H(v_n - v_n(\tau)) \|_{L^1} \\
\leq \| u_n(\tau) \|_{L^2} \| H(v_n - v_n(\tau)) \|_{L^2} \\
\leq (\gamma C + \alpha) \| u_n(\tau) \|_{L^2} \| (u_n - u_n)(\tau) \|_{L^2} \to 0 \quad (n \to \infty).
\]
\[
M_4 = \| (u_n - u_n)(\tau)H v_n(\tau) \|_{L^1} \\
\leq \| (u_n - u_n)(\tau) \|_{L^2} \| H v_n \|_{L^2} \to 0 \quad (n \to \infty).
\]

Observe (124) and (125). Consider each convergent term \( M_i \) evaluated above. Then by the Lebesgue’s convergence theorem, we obtain
\[
\sup_{0 < t < T} \| u_n(t) - \left\{ e^{t\overline{a}} - \chi \int_0^t e^{(t-\tau)L} \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \, d\tau \right\} \|_{L^2} = 0. \tag{126}
\]
ii) Evaluation of \( \| u(t) - \left\{ e^{t\Delta} - \chi \int_0^t e^{(t-\tau)\Delta} \partial_x(u(\tau) \eta \partial_x v(\tau)) \, d\tau \right\} \|_{L^\infty} \):

By the same discussion as i) (cf. (124), (125) and each convergent term \( M \) above),

\[
\| u(t) - \left\{ e^{t\Delta} - \chi \int_0^t e^{(t-\tau)\Delta} \partial_x(u(\tau) \eta \partial_x v(\tau)) \, d\tau \right\} \|_{L^\infty} \\
\leq \| (u - u_n) (t) \|_{L^\infty} \\
+ \chi \int_0^t \| e^{(t-\tau)\Delta} (\partial_x(u(\tau) \eta \partial_x v(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^\infty} \, d\tau \\
\to 0 \quad (n \to \infty).
\]

Hence we have the following:

\[
\sup_{0 < t < T} \| u(t) - \left\{ e^{t\Delta} - \chi \int_0^t e^{(t-\tau)\Delta} \partial_x(u(\tau) \eta \partial_x v(\tau)) \, d\tau \right\} \|_{L^\infty} = 0. \tag{127}
\]

iii) Evaluation of \( t^{\frac{3}{2}} \| \partial_x[u(t) - \left\{ e^{t\Delta} - \chi \int_0^t e^{(t-\tau)\Delta} \partial_x(u(\tau) \eta \partial_x v(\tau)) \, d\tau \right\}] \|_{L^2} \):

By (123) we see

\[
\| \partial_x[u(t) - \left\{ e^{t\Delta} - \chi \int_0^t e^{(t-\tau)\Delta} \partial_x(u(\tau) \eta \partial_x v(\tau)) \, d\tau \right\}] \|_{L^2} \\
\leq \| \partial_x(u - u_n) (t) \|_{L^2} \\
+ \chi \int_0^t \| \partial_x e^{(t-\tau)\Delta} \partial_x(u(\tau) \eta \partial_x v(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^2} \, d\tau \\
\leq \| \partial_x(u - u_n) (t) \|_{L^2} \\
+ \chi \int_0^t \frac{\lambda^2}{(t - \tau)^{\frac{5}{2}}} \| \partial_x(u(\tau) \eta \partial_x v(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^2} \, d\tau. \tag{128}
\]

Here, the integrand of the right-hand side of (128) is evaluated as follows (cf. (125)):

\[
\| \partial_x(u(\tau) \eta \partial_x v(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^2} \\
= \| \partial_x(u(\tau) \eta \partial_x v(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) - u_n H v_n(\tau) \|_{L^2} \\
\leq \| \partial_x(u(\tau) \eta \partial_x v(\tau)) - \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) \|_{L^2} \\
+ \| \partial_x(u_n(\tau) \eta \partial_x v_n(\tau)) - u_n H v_n(\tau) \|_{L^2} \\
=: N_1 + N_2 + N_3 + N_4. \tag{129}
\]

By the inequality (55) and the convergences (119) – (121), each \( N_i \) is evaluated as follows (cf. \( M_1 - M_4 \) : for any \( \tau \in (0, T) \),

\[
N_1 = \| \partial_x(u(\tau) \eta \partial_x(v_\tau - v_n(\tau)) \|_{L^2} \\
\leq \| \partial_x(u(\tau)) \|_{L^2} \| \eta \|_{L^\infty} \| \partial_x(v_\tau - v_n(\tau)) \|_{L^\infty} \\
\leq C \| \partial_x(u(\tau)) \|_{L^2} \| \eta \|_{L^\infty} \| (u_\tau - u_n(\tau)) \|_{L^2} \to 0 \quad (n \to \infty).
\]
\[ N_2 = \| \partial_x (u_s - u_n)(\tau) \eta \partial_x v_n(\tau) \|_{L^2} \]
\[ \leq \| \partial_x (u_s - u_n)(\tau) \|_{L^2} \| \eta \|_{L^\infty} \| \partial_x v_n(\tau) \|_{L^\infty} \to 0 \ (n \to \infty). \]
\[ N_3 = \| u_s(\tau) H(v_s - v_n)(\tau) \|_{L^2} \]
\[ \leq \| u_s(\tau) \|_{L^\infty} \| H(v_s - v_n)(\tau) \|_{L^2} \]
\[ \leq (\gamma C + \alpha) \| u_s(\tau) \|_{L^\infty} \| (u_s - u_n)(\tau) \|_{L^2} \to 0 \ (n \to \infty). \]
\[ N_4 = \| (u_s - u_n)(\tau) H v_n(\tau) \|_{L^2} \]
\[ \leq \| (u_s - u_n)(\tau) \|_{L^\infty} \| H v_n \|_{L^2} \to 0 \ (n \to \infty). \]

Combine (128) and (129) with the preceding estimates for \( N_i \). Then by the Lebesgue’s convergence theorem, we obtain

\[ \sup_{0 < t < T} t^{\frac{1}{2}} \| \partial_x [u_s(t) - \left\{ e^{t \Delta} \bar{u} - \int_0^t e^{(t-\tau)\Delta} \partial_x (u_s(\tau) \eta \partial_x v_s(\tau)) \, d\tau \right\}] \|_{L^2} = 0. \quad (130) \]

Observe the Banach spaces \( X \) and \( Y \) given by (56) and (57), respectively. Then, by the convergences (119) – (121) and the definition (122) together with the results (126), (127) and (130), we conclude that \( (u_s, v_s) \) is a mild solution of (GKS) on (0, T) satisfying \( u_s \in X, v_s \in Y \). This completes the proof of the existence part of Theorem 5.3. 

### 5.5 Proof of the uniqueness part of Theorem 5.3

In this subsection, we give a proof of the uniqueness part of Theorem 5.3, that is, we prove that \( (u_s, v_s) \) is the unique mild solution of (GKS) on (0, T) with the same initial state \( u(x, 0) = \tilde{u}(x, 0) = \pi(x) \). Let \( (u, v) \) and \( (\tilde{u}, \tilde{v}) \) be mild solutions of (GKS) on (0, T). Then it holds that

\[ \| (u - \tilde{u})(t) \|_{L^2} \]
\[ \leq \chi \| \int_0^t e^{(t-\tau)\Delta} \left( \partial_x (u(\tau) \eta \partial_x v(\tau)) - \partial_x (\tilde{u}(\tau) \eta \partial_x \tilde{v}(\tau)) \right) \, d\tau \|_{L^2} \]
\[ \leq \chi \int_0^t \frac{k}{(t - \tau)^\frac{3}{2}} \left( \| \partial_x u(\tau) \eta \partial_x (v - \tilde{v}(\tau)) \|_{L^1} + \| \partial_x (u - \tilde{u})(\tau) \eta \partial_x \tilde{v}(\tau) \|_{L^1} \right) \| u(\tau) \|_{H^1} \| H(\tilde{v}(\tau)) \|_{L^1} \, d\tau. \quad (131) \]

In (131), we use the first integral equation in Definition 5.2 for the first inequality, (53) and the same discussion as that in (101) for the second inequality (cf. (63) – (65)). As well as \( K_1 - K_4 \) in the proof of the existence part of Theorem 5.3, we obtain the following estimates.

\[ \| \partial_x u(\tau) \eta \partial_x (v - \tilde{v}(\tau)) \|_{L^1} \leq C \tau^{-\frac{1}{2}} \lambda_2 \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2}. \quad (132) \]
\[ \| \partial_x (u - \tilde{u})(t) \|_{L^1} \leq C \tau^{-\frac{1}{2}} x_s \lambda_2 \sup_{0 < t < T} t^\frac{1}{2} \| \partial_x (u - \tilde{u})(t) \|_{L^2}. \]  
(133)

\[ \| u(\tau) H(v - \tilde{v})(\tau) \|_{L^1} \leq (\gamma C + \alpha) x_s \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2}. \]  
(134)

\[ \| (u - \tilde{u})(\tau) H \tilde{v}(\tau) \|_{L^1} \leq (\gamma C + \alpha) x_s \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2}. \]  
(135)

By substituting these estimates (132) – (135) into (131), we have

\[
\| (u - \tilde{u})(t) \|_{L^2} \leq \chi \int_0^t \frac{k}{(t - \tau)^{\frac{1}{2}}} \left( \tau^{-\frac{1}{2}} x_s \lambda_2 C \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} \right.
+ \tau^{-\frac{1}{2}} x_s \lambda_2 C \sup_{0 < t < T} t^\frac{1}{2} \| \partial_x (u - \tilde{u})(t) \|_{L^2}
\left. + 2(\gamma C + \alpha) x_s \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} \right) d\tau
\leq \left( \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + \sup_{0 < t < T} t^\frac{1}{2} \| \partial_x (u - \tilde{u})(t) \|_{L^2} \right)
\times \left( \chi k B \left( \frac{3}{4}, \frac{1}{2} \right) x_s \lambda_2 C T^\frac{1}{2} + \frac{8}{3} \chi k (\gamma C + \alpha) x_s T^\frac{1}{2} \right). \]

(136)

Define

\[ \hat{D}_1 := \max \{ \chi k B \left( \frac{3}{4}, \frac{1}{2} \right) \lambda_2 C, \frac{8}{3} \chi k (\gamma C + \alpha) \}. \]

Then, from (136) we obtain

\[ \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} \leq \left( \sup_{0 < t < T} \| u - \tilde{u} \|_{L^2} + \sup_{0 < t < T} t^\frac{1}{2} \| \partial_x (u - \tilde{u})(t) \|_{L^2} \right) (\hat{D}_1 x_s T^\frac{1}{2} + \hat{D}_1 x_s T^\frac{1}{2}). \]  
(137)

Nextly, the inequality (52) and the same argument as that in (131) – (135), imply the following:

\[ \| (u - \tilde{u})(t) \|_{L^\infty} \leq \chi \int_0^t e^{(t-\tau)L} \left( \partial_x (u(\tau) \eta \partial_x v(\tau)) - \partial_x (\tilde{u}(\tau) \eta \partial_x \tilde{v}(\tau)) \right) d\tau \|_{L^\infty} \]
\[ \leq \chi \int_0^t \frac{l}{(t - \tau)^{\frac{1}{2}}} \| \partial_x (u(\tau) \eta \partial_x v(\tau)) - \partial_x (\tilde{u}(\tau) \eta \partial_x \tilde{v}(\tau)) \|_{L^1} d\tau \]
\[ \leq \chi \int_0^t \frac{l}{(t - \tau)^{\frac{1}{2}}} \left( \tau^{-\frac{1}{2}} x_s \lambda_2 C \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2}
\right.
\left. + \tau^{-\frac{1}{2}} x_s \lambda_2 C \sup_{0 < t < T} t^\frac{1}{2} \| \partial_x (u - \tilde{u})(t) \|_{L^2}
\right)
\left. + 2(\gamma C + \alpha) x_s \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} \right) d\tau
\leq \left( \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + \sup_{0 < t < T} t^\frac{1}{2} \| \partial_x (u - \tilde{u})(t) \|_{L^2} \right)
\times \left( \chi L x_s \lambda_2 C B \left( \frac{1}{2}, \frac{1}{2} \right) + 4 \chi l (\gamma C + \alpha) x_s t^\frac{1}{2} \right)
\leq \left( \sup_{0 < t < T} \| u - \tilde{u} \|_{L^2} + \sup_{0 < t < T} t^\frac{1}{2} \| \partial_x (u - \tilde{u})(t) \|_{L^2} \right)
\times \left( \chi L x_s \lambda_2 C \pi + 4 \chi l (\gamma C + \alpha) x_s T^\frac{1}{2} \right). \]  
46
Finally, by the first integral equation in Definition 5.2 and the norm inequality (51), we have

\[
\sup_{0 < t < T} \|(u - \bar{u})(t)\|_{L^\infty} \leq \left( \sup_{0 < t < T} \|u - \bar{u}\|_{L^2} + \sup_{0 < t < T} t^{\frac{1}{2}} \|\partial_x(u - \bar{u})\|_{L^2} \right) (\hat{D}_2 z_* + \hat{D}_2 x_* T^{\frac{1}{2}}),
\]

where

\[
\hat{D}_2 := \max \{ \chi l \lambda_2 C \pi, 4 \chi l (\gamma C + \alpha) \}.
\]

Finally, by the first integral equation in Definition 5.2 and the norm inequality (51), we have

\[
\|\partial_x (u - \tilde{u})(t)\|_{L^2} \leq \chi \int_0^t \|\partial_x e^{(t-\tau)L} \left( \partial_x (u(\tau) \eta \partial_x v(\tau)) - \partial_x (\tilde{u}(\tau) \eta \partial_x \tilde{v}(\tau)) \right) \|_{L^2} d\tau
\]

\[
\leq \chi \int_0^t \frac{\lambda_1^2}{(t - \tau)^{\frac{1}{2}}} \|\partial_x (u(\tau) \eta \partial_x v(\tau)) - \partial_x (\tilde{u}(\tau) \eta \partial_x \tilde{v}(\tau))\|_{L^2} d\tau. \tag{139}
\]

Consider (101) with \((u_n, v_n)\) and \((u_{n-1}, v_{n-1})\) replaced by \((u, v)\) and \((\tilde{u}, \tilde{v})\), respectively. Then, as in (101), we obtain

\[
\|\partial_x (u(\tau) \eta \partial_x v(\tau)) - \partial_x (\tilde{u}(\tau) \eta \partial_x \tilde{v}(\tau))\|_{L^2}
\]

\[
\leq \|\partial_x u(\tau) \eta \partial_x (v - \tilde{v})(\tau)\|_{L^2} + \|\partial_x (u - \tilde{u})(\tau) \eta \partial_x \tilde{v}(\tau)\|_{L^2}
\]

\[
+ \|u(\tau) H (v - \tilde{v})(\tau)\|_{L^2} + \|(u - \tilde{u})(\tau) H \tilde{v}(\tau)\|_{L^2}. \tag{140}
\]

The same argument as that in (132) – (135) leads to the following:

\[
\|\partial_x (u(\tau) \eta \partial_x (v - \tilde{v})(\tau))\|_{L^2} \leq C \tau^{-\frac{1}{2}} z_* \lambda_2 \sup_{0 < t < T} \|(u - \tilde{u})(t)\|_{L^2}, \tag{141}
\]

\[
\|\partial_x (u - \tilde{u})(\tau) \eta \partial_x \tilde{v}(\tau)\|_{L^2} \leq C \tau^{-\frac{1}{2}} x_* \lambda_2 \sup_{0 < t < T} t^{\frac{1}{2}} \|\partial_x (u - \tilde{u})\|_{L^2}, \tag{142}
\]

\[
\|u(\tau) H (v - \tilde{v})(\tau)\|_{L^2} \leq (\gamma C + \alpha) y_* \sup_{0 < t < T} \|(u - \tilde{u})(t)\|_{L^2}, \tag{143}
\]

\[
\|(u - \tilde{u})(\tau) H \tilde{v}(\tau)\|_{L^2} \leq (\gamma C + \alpha) x_* \sup_{0 < t < T} \|(u - \tilde{u})(t)\|_{L^2}. \tag{144}
\]

Hence, the estimates (139) – (144) yield

\[
\|\partial_x (u - \tilde{u})(t)\|_{L^2}
\]

\[
\leq \chi \lambda_1^2 \left( \lambda_2 C B \left( \frac{1}{2}, \frac{1}{2} \right) (x_* + z_*) + (\gamma C + \alpha) x_* t^{\frac{1}{2}} + (\gamma C + \alpha) y_* t^{\frac{1}{2}} \right)
\]

\[
\times \left( \sup_{0 < t < T} \|(u - \tilde{u})(t)\|_{L^2} + \sup_{0 < t < T} \|(u - \tilde{u})(t)\|_{L^\infty} + \sup_{0 < t < T} t^{\frac{1}{2}} \|\partial_x (u - \tilde{u})(t)\|_{L^2} \right).
\]

Set

\[
\hat{D}_3 := \max \{ \chi \lambda_1^2 \lambda_2 C \pi, \chi \lambda_1^2 (\gamma C + \alpha) \}.
\]
Then, it holds that
\[
\sup_{0 < t < T} t^{\frac{3}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} \\
\leq (\tilde{D}_3(x_* + z_*) T^{\frac{1}{4}} + \tilde{D}_3(x_* + y_*) T^1) \\
\times (\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^\infty} + \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2}) \\
+ \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2}.
\]
(145)

Lastly, we define \( \tilde{D}_* = \tilde{D}_* (\chi, \alpha, \gamma, \lambda_1, \lambda_2) \) by
\[
\tilde{D}_* := \max\{ \tilde{D}_1, \tilde{D}_2, \tilde{D}_3 \}.
\]

Then, from (137), (138) and (145), we obtain the following three estimates numbered (146), (147) and (148), respectively:
\[
\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} \\
\leq (\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} ) (\tilde{D}_* T^{\frac{1}{4}} + \tilde{D}_* T^1),
\]
(146)
\[
T^{\frac{1}{4}} \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^\infty} \\
\leq (\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} ) (\tilde{D}_* T^{\frac{1}{4}} + \tilde{D}_* T^1),
\]
(147)
\[
\sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} \\
\leq (\tilde{D}_* (x_* + z_*) T^{\frac{1}{4}} + \tilde{D}_* (x_* + y_*) T^1) \\
\times (\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^\infty} + \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} ) \\
\times (\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + T^{\frac{1}{4}} \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^\infty} + \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} ).
\]
(148)

By adding the three inequalities (99) – (101), we have
\[
\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + T^{\frac{1}{4}} \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^\infty} + \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} \\
\leq (\tilde{D}_* (3z_* + x_*) T^{\frac{1}{4}} + \tilde{D}_* (3x_* + y_*) T^1) \\
\times (\sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^2} + T^{\frac{1}{4}} \sup_{0 < t < T} \| (u - \tilde{u})(t) \|_{L^\infty} + \sup_{0 < t < T} t^{\frac{1}{4}} \| \partial_x (u - \tilde{u})(t) \|_{L^2} ).
\]
(149)
Substitute (89), i.e., the values $z_* = x_*$ and $y_* = x_*T^{-\frac{1}{2}}$, into (149). Then, we obtain

$$
\begin{align*}
\sup_{0<t<T} \|(u - \tilde{u})(t)\|_{L^2} + T^{\frac{1}{4}} \sup_{0<t<T} \|(u - \tilde{u})(t)\|_{L^\infty} &+ T^{\frac{1}{4}} \sup_{0<t<T} t^{\frac{1}{2}} \|\partial_x(u - \tilde{u})(t)\|_{L^2} \\
\leq \hat{D}_s T^{\frac{1}{4}} x_*(4 + T^{\frac{1}{2}} + 3T^{\frac{3}{4}}) \times \left( \sup_{0<t<T} \|(u - \tilde{u})(t)\|_{L^2} + T^{\frac{1}{4}} \sup_{0<t<T} \|(u - \tilde{u})(t)\|_{L^\infty} \\
+ \sup_{0<t<T} t^{\frac{1}{2}} \|\partial_x(u - \tilde{u})(t)\|_{L^2} \right). \quad (150)
\end{align*}
$$

By (150), we see that under the condition

$$
\hat{D}_s T^{\frac{1}{4}} x_*(4 + T^{\frac{1}{2}} + 3T^{\frac{3}{4}}) < 1, \quad (151)
$$

the following must hold:

$$
\sup_{0<t<T} \|(u - \tilde{u})(t)\|_{L^2} + T^{\frac{1}{4}} \sup_{0<t<T} \|(u - \tilde{u})(t)\|_{L^\infty} + T^{\frac{1}{4}} \sup_{0<t<T} t^{\frac{1}{2}} \|\partial_x(u - \tilde{u})(t)\|_{L^2} = 0. \quad (152)
$$

Here, since $x_*$ depends on $T$, we have to check whether (151) holds for small $T > 0$. For this aim, we rewrite (151) with $x_*$ replaced by the value (88), and obtain

$$
\hat{D}_s \frac{1 - \sqrt{1 - 4D_s(T^{\frac{1}{4}} + T^{\frac{3}{4}})D_0}}{2D_s(1 + T^{\frac{1}{2}})}(4 + T^{\frac{1}{2}} + 3T^{\frac{3}{4}}) < 1. \quad (153)
$$

The condition (153), i.e. (151) holds for small $T > 0$. Notice that $T = T(D_0, D_*, \hat{D}_s) = T(\chi, \alpha, \gamma, \pi, \lambda_1, \lambda_2)$. The equation (152) tells us that $u = \tilde{u}$, and $v = \alpha(\gamma - H)^{-1}u = \alpha(\gamma - H)^{-1}\tilde{u} = \tilde{v}$. ■

### 5.6 Proof of Theorem 5.4

Notice that $S(\mathbb{R})$ is dense in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Consider the inequalities (55) and the construction of $(u_*, v_*)$ performed in Section 5.4. Then, by standard density arguments, Theorem 5.4 is proved to hold. ■
A Appendix 1

In this appendix, we shall give the detailed proofs of (14), (13) and (18) given in Chapter 3.

A.1 Proof of (14)

In this section, we show the detailed proof of (14).

\[ d\left( \tilde{u}(X(s), s) e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau} \right) \]

\[ = d\left( \tilde{u}(X(s), s) \right) \cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau} \]

\[ + \tilde{u}(X(s), s) \cdot d\left( e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau} \right) \]

\[ = \left[ (\tilde{u}_x(X(s), s) \cdot (-a\tilde{v}_x(X(s), s)) ds + \sqrt{2} \tilde{u}_x(X(s), s) dB_s \right] \]

\[ + \tilde{u}_{xx}(X(s), s) ds + \tilde{u}_s(X(s), s) ds \cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau} \]

\[ + \tilde{u}(X(s), s) \cdot (-a\tilde{v}_{xx}(X(s), s)) \cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau} ds \]

\[ = \left[ (\tilde{u}_x(X(s), s) \cdot (-a\tilde{v}_x(X(s), s)) ds + \sqrt{2}\tilde{u}_x(X(s), s) dB_s \right] \]

\[ + \tilde{u}_{xx}(X(s), s) ds - \tilde{u}_{xx}(X(s), s) ds + a\tilde{u}_x(X(s), s) \cdot \tilde{v}_x(X(s), s) ds \]

\[ + a\tilde{u}(X(s), s) \tilde{v}_{xx}(X(t), s) ds \cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau} \]

\[ + \tilde{u}(X(s), s) \cdot (-a\tilde{v}_{xx}(X(s), s)) \cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau} ds \]

\[ = \sqrt{2}\tilde{u}_x(X(s), s) dB_s \cdot e^{-\int_t^s a\tilde{v}_{xx}(X(\tau), \tau) d\tau}, \]

where we have used the differential calculus of a composite function, (10) and the generalized Itô’s formula (cf. Prop. 3.2) for the second equation, and (4.1) for the third equation. Also we have used the fact that the term of local time \( \phi_1 \) in (10) vanishes for the functional \( \tilde{u} \) satisfying \( \tilde{u}_x(L_1, t) = \tilde{u}_x(L_2, t) = 0 \) (cf. Section IV-7 of [2], and [7] and references therein). \( \blacksquare \)
A.2 Proof of (13)

As was done for the proof of (14), we can prove (13). Let us consider a functional of \( \tilde{v}(Y(s), s) e^{-\gamma \int_t^s d\tau} \). By using (9.2) and (11) instead of (9.1) and (10) in A.1, we have the following relation:

\[
d \left( \tilde{v}(Y(s), s) e^{-\gamma \int_t^s d\tau} \right) = d \left( \tilde{v}(Y(s), s) \right) \cdot e^{-\gamma \int_t^s d\tau} + \tilde{v}(Y(s), s) \cdot d \left( e^{-\gamma \int_t^s d\tau} \right)
\]

\[
= \left[ \tilde{v}_x(Y(s), s) \cdot 0 ds + \sqrt{2} \tilde{v}_x(Y(s), s) dB_s + \tilde{v}_{xx}(Y(s), s) ds \\
+ \tilde{v}_s(Y(s), s) ds \right] \cdot e^{-\gamma \int_t^s d\tau} + \tilde{v}(Y(s), s) \cdot (-\gamma) \cdot e^{-\gamma \int_t^s d\tau} ds
\]

\[
= \left[ \sqrt{2} \tilde{v}_x(Y(s), s) dB_s + \tilde{v}_{xx}(Y(s), s) ds - \tilde{v}_{xx}(Y(s), s) ds + \gamma \tilde{v}(Y(s), s) ds \\
- \alpha \tilde{u}(Y(s), s) ds \right] \cdot e^{-\gamma \int_t^s d\tau} + \tilde{v}(Y(s), s) \cdot (-\gamma) \cdot e^{-\gamma \int_t^s d\tau} ds
\]

\[
= \left( \sqrt{2} \tilde{v}_x(Y(s), s) dB_s - \alpha \tilde{u}(Y(s), s) ds \right) e^{-\gamma \int_t^s d\tau}.
\]

By integrating both sides from \( t \) to \( T \), we have

\[
\int_t^T d \left( \tilde{v}(Y(s), s) \right) e^{-\gamma \int_t^s d\tau} ds
\]

\[
= \int_t^T \left( \sqrt{2} \tilde{v}_x(Y(s), s) dB_s - \alpha \tilde{u}(Y(s), s) ds \right) e^{-\gamma \int_t^s d\tau}.
\]

It follows that

\[
\tilde{v}(Y(T), T) e^{-\gamma \int_t^T d\tau} - \tilde{v}(Y(t), t) e^{-\gamma \int_t^t d\tau}
\]

\[
= \sqrt{2} \int_t^T e^{-\gamma \int_t^s d\tau} \tilde{v}_x(Y(s), s) dB_s - \int_t^T \alpha \tilde{u}(Y(s), s) e^{-\gamma \int_t^s d\tau} ds.
\]

After putting in order, we take the expectation \( E \) in probability measure \( P \), then we have
\[ E[\tilde{v}(Y(T), T)e^{-\gamma(T-t)} - \tilde{v}(Y(t), t) \mid Y(t) = x] \]
\[ = E[\sqrt{2} \int_t^T e^{-\gamma(s-t)}\tilde{v}_x(Y(s), s)dB_s \mid Y(t) = x] \]
\[ - E[\int_t^T \alpha \tilde{v}(Y(s), s)e^{-\gamma(s-t)}ds \mid Y(t) = x]. \]

Since the first term on the right-hand side equals to zero, and \( v(x, T) \) equals to \( \overline{v}(x) \), we have (13). \( \blacksquare \)

### A.3 Proof of (18)

Finally we shall prove the inequality (18). By (17),

\[ \tilde{u}(Y(\tau), \tau) = \tilde{u}(Y(\tau), T - (T - \tau)) \geq \inf_{x \in I} \overline{u} \cdot e^{-(T-\tau)aM} \]

holds. Thus for \( \gamma \neq aM \), we have

\[ II = E \left[ \int_{T-s}^T \alpha \tilde{u}(Y(\tau), \tau)e^{-\gamma(\tau-T+s)} d\tau \mid Y(T - s) = x \right] \]
\[ \geq \alpha \int_{T-s}^T \inf_{x \in I} \overline{u} \cdot e^{-(T-\tau)aM} e^{-\gamma(\tau-T+s)} d\tau \]
\[ = \alpha \int_{T-s}^T \overline{u} \cdot e^{-TaM+\gamma T-\gamma s} \int_{T-s}^T e^{(aM-\gamma)\tau} d\tau \]
\[ = \alpha \int_{T-s}^T \overline{u} \cdot e^{-TaM+\gamma T-\gamma s} \frac{1}{aM - \gamma} \left( e^{(aM-\gamma)T} - e^{(aM-\gamma)(T-s)} \right) \]
\[ = \alpha \int_{T-s}^T \overline{u} \cdot e^{-TaM+\gamma T-\gamma s} \frac{1}{aM - \gamma} \left( e^{(aM-\gamma)(T-s)} - e^{(aM-\gamma)(-s)} \right) \]
\[ = \alpha \int_{T-s}^T \overline{u} \cdot e^{-\gamma s} \frac{1}{aM - \gamma} \left( (1 - e^{(aM-\gamma)(-s)}) \right) \]
\[ = \alpha \int_{T-s}^T \overline{u} \cdot e^{-\gamma s} \frac{1}{aM - \gamma} \left( e^{-(aM-\gamma)s} \right). \]

Also for \( \gamma = aM \), we have

\[ II = E \left[ \int_{T-s}^T \alpha \tilde{u}(Y(\tau), \tau)e^{-\gamma(\tau-T+s)} d\tau \mid Y(T - s) = x \right] \]
\[ \geq \alpha \int_{T-s}^T \inf_{x \in I} \overline{u} \cdot e^{-(T-\tau)aM} e^{-\gamma(\tau-T+s)} d\tau \]
\[ = \alpha \int_{T-s}^T \overline{u} \cdot e^{-\gamma s} \int_{T-s}^T d\tau \]
\[ = \alpha \int_{T-s}^T \overline{u} \cdot e^{-\gamma s}. \]
Appendix 2

In this appendix, we shall present the proofs of Propositions 4.3 and 4.6 given in Chapter 4.

B.1 Proof of Proposition 4.3.

We use an eigenfunction expansion method. We set

\[ w(x, t) := \sum_{n=0}^{\infty} w_n(x, t) = \sum_{n=0}^{\infty} T_n(t)X_n(x), \]  

(154)

where

\[ X_n(x) = \cos \lambda_n(x - a), \quad \lambda_n = \frac{n \pi}{b - a}. \]  

(155)

Step 1. We set

\[ z(x, t) := \sum_{n=0}^{\infty} z_n(t)X_n(x). \]  

(156)

From (156), we have the following equation:

\[ \int_{a}^{b} z(x, t) \cos \lambda_m(x - a) \, dx = \int_{a}^{b} \sum_{n=0}^{\infty} z_n(t) \cos \lambda_n(x - a) \cos \lambda_m(x - a) \, dx. \]  

(157)

Note that if \( n \neq m \), then it holds that

\[ \int_{a}^{b} \cos \lambda_n(x - a) \cos \lambda_m(x - a) \, dx = 0, \]

and in the case \( n = m \neq 0 \), then it holds that

\[ \int_{a}^{b} \cos \lambda_n(x - a) \cos \lambda_n(x - a) \, dx = \frac{b - a}{2}. \]

By substituting these formulas into (157), we obtain

\[ \int_{a}^{b} z(x, t) \cos \lambda_m(x - a) \, dx = \frac{b - a}{2} z_n(t) \, (n \geq 1), \]

namely,

\[ z_n(t) = \frac{2}{b - a} \int_{a}^{b} z(x, t) \cos \lambda_n(x - a) \, dx \, (n \geq 1). \]

In the case \( n = m = 0 \), it holds that \( \int_{a}^{b} \cos \lambda_0(x - a) \cos \lambda_0(x - a) \, dx = b - a \),

thus it follows that

\[ z_0(t) = \frac{1}{b - a} \int_{a}^{b} z(x, t) \, dx. \]
Step 2. From (154) and (156), we know that (22) given in Proposition 4.3 means
\[ T'_n(t)X_n(x) = T_n(t)X''_n(x) + z_n(t)X_n(x). \] (158)
For \( n \geq 1 \), (158) is transformed to
\[ T'_n(t) + \lambda^2_n T_n(t) = z_n(t). \] (159)
On the other hand, from (154) and (155), we notice that
\[ \bar{w}(x) = \sum_{n=0}^{\infty} T_n(0) \cos \lambda_n(x - a). \]
With the Fourier series expansion of \( \bar{w}(x) \), we have
\[ T_n(0) = \frac{2}{b-a} \int_a^b \bar{w}(x) \cos \lambda_n(x - a) \, dx \quad (n \geq 1) \]
and
\[ T_0(0) = \frac{1}{b-a} \int_a^b \bar{w}(x) \, dx. \]
For \( n = 0 \), (158) means
\[ T'_0(t) = z_0(t). \] (160)
Step 3. Let us solve the differential equations (159) and (160). By a standard method, we have the following solutions:
\[ T_n(t) = e^{-\lambda^2_n t} \left( T_n(0) + \int_0^t z_n(\tau) e^{\lambda^2_n \tau} \, d\tau \right) \quad (n \geq 1), \] (161)
and for \( n = 0 \),
\[ T_0(t) = T_0(0) + \int_0^t z_n(\tau) \, d\tau. \]
Notice that (161) is correct in the case \( n = 0 \).

Step 4. Combined (154), (155) with (161), we have proved Proposition 4.3.

**B.2 Proof of Proposition 4.6.**

We will show that for any \( \epsilon > 0 \), there exists \( t_0 > 0 \) such that \( F(t) < \epsilon \) for all \( t \geq t_0 \).
We set \( \delta = \frac{k}{2l} \epsilon \). On the other hand, because it is assumed that \( \lim_{t \to \infty} G(t) = 0 \), we know that there exists \( t_1 > 0 \) such that \( G(t) < \delta \) for all \( t \geq t_1 \). From the given assumption, we have
\[ F'(t) \leq -kF(t) + l\delta = -k \left\{ F(t) - \frac{l\delta}{k} \right\}. \]
We set \( X(t) = F(t) - \frac{l\delta}{k} \). Then it follows that

\[ X'(t) \leq -kX(t). \]

By Proposition 4.5, we obtain the following inequality:

\[ X(t) \leq X(t_1)e^{-k(t-t_1)}. \]

Therefore, we have

\[ F(t) \leq \{ F(t_1) - \frac{l\delta}{k} \}e^{-k(t-t_1)} + \frac{l\delta}{k} \leq F(t_1)e^{-k(t-t_1)} + \frac{l\delta}{k}. \]  \hspace{1cm} (162)

Since we can choose \( t_0(> t_1) \) such that

\[ F(t_1)e^{-k(t_0-t_1)} < \frac{\epsilon}{2}, \]

the inequality (162) means \( F(t) < \epsilon \) for all \( t > t_0 \). ■

C Appendix 3

Figures of numerical examples are drawn by making use of the mathematical software, Mathematica. The following is the source code of Mathematica through which we draw the 3-dimensional graphs in Figure 6.

```mathematica
In[1]:=

usol = NDSolve[{
D[u[t, x], t] == D[u[t, x], x, x] - 5/4* D[u[t, x] * D[v[t, x], x], x],
D[v[t, x], t] == D[v[t, x], x, x] + 2 u[t, x] - 3 v[t, x],

u[0, x] == 3 - Cos[2 x], v[0, x] == 3, u[0,1][t, -10] == 0, u[0,1][t, 10] == 0,

v[0,1][t, -10] == 0, v[0,1][t, 10] == 0, {u, v}, {t, 0, 6}, {x, -10, 10}}

Plot3D[u[t, x] /. usol, {t, 0, 6}, {x, 0, Pi}, PlotRange -> {0, 6}, {0, Pi}, {0, 5}],
PlotPoints ->50, AxesLabel -> {t, x}]

Plot3D[v[t, x] /. usol, {t, 0, 6}, {x, 0, Pi}, PlotRange -> {0, 6}, {0, Pi}, {0, 5}],
PlotPoints ->50, AxesLabel -> {t, x}]
```

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References


[9] Cieślak, T., Laurençot, P. and Morales-Rodrigo, C., Global existence and convergence to steady states in a chemorepulsion system, Parabolic and Navier-Stokes equations Part1, 81 (2008), 105-117.


